Bieri–Strebel Invariants and Bergman Fans

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1 Introduction

This essay concentrates on two independently defined but closely related concepts, namely those of tropical varieties and Bieri–Strebel invariants. The former of these is highly geometric in nature, and it joins the theory on valuations of rings with that on algebraic varieties. The latter one was primarily motivated by group theory – in particular, it helped to solve a problem on whether a metabelian group is finitely presentable.

The earliest concept described in this essay was introduced by George Bergman in his 1971 paper [2], where it was called a “logarithmic limit set”, now known as a Bergman complex. It was defined as a closed subset of the unit sphere in a finite-dimensional real vector space. Using the formalism of tropical geometry, which will be introduced in Chapter 3, it is now known that a Bergman complex is a polyhedral set, i.e. it is a projection of a union of convex polytopes to the unit sphere.

During the past 20–30 years, a new branch of mathematics – tropical geometry – has appeared [4]. In this setting, the set \( \mathbb{R} \cup \{\infty\} \) was given the structure of a semiring, and valuations (maps from an arbitrary integral domain to this semiring which almost preserve the semiring structure) were considered. This theory was then applied to the formalism of algebraic geometry, and a new type of polyhedral sets, called tropical varieties, was introduced. This construction was almost the same as – but in a sense more general than – the original definition of Bergman complexes. In particular, Bergman complexes describe the behaviour of tropical varieties “at infinity”, and they are independent of the valuation chosen (whereas a tropical variety depends on the valuation).

A seemingly unrelated concept was introduced by Robert Bieri and Ralph Strebel in their 1980 paper [3], and is now known as a Bieri–Strebel invariant. These invariants were defined in order to solve a problem in group theory: namely, the criterion for a metabelian group (a group whose derived subgroup is abelian) to have a finite presentation. During the development of tropical geometry, it became clear that Bieri–Strebel invariants are closely related to tropical varieties. The main aim of this essay is to explain this relation.

The structure of the essay is as follows. Chapter 2 of the essay introduces the concepts of Laurent polynomial rings, characters of abelian groups, valuations of rings, and combines these concepts to define weights of polynomials, which will be important in later chapters. Chapter 3 introduces the notion of tropical varieties and briefly discusses their geometric properties. In Chapter 4 Bieri–Strebel invariants are defined and several basic properties on them are noted, and their relation to tropical varieties is discussed. Chapter 5 provides a historical motivation of Bieri–Strebel invariants – in particular, their relation to metabelian groups.
1.1 Notation

We will write $\mathbb{R}_\infty$ for the totally ordered set $\mathbb{R} \cup \{\infty\}$, where the order is given by the usual total order on $\mathbb{R}$, and $\infty$ is the maximal element of $\mathbb{R}_\infty$.

The standard inner product on the vector space $\mathbb{R}^n$ will be denoted by $\langle \cdot , \cdot \rangle$, and $\| \cdot \|$ will denote the induced norm. Also, given a finite subset $F \subseteq \mathbb{R}^n$ and a vector $\chi \in \mathbb{R}^n$, we will write

$$\langle \chi , F \rangle := \min\{ \langle \chi , y \rangle \mid y \in F \}.$$ 

All rings considered in the essay are commutative with a 1.

Whenever groups are considered, symbol $e$ will be used to denote the identity element of a group, and so $\{e\}$ will denote the trivial group. For group elements $a$ and $b$, $[a,b]$ will denote their commutator $aba^{-1}b^{-1}$. 

2 Basic setup

This chapter introduces concepts of characters of abelian groups and valuations of integral domains, as well as group rings and weights over them. Definitions and a proposition introduced in this chapter are essential for the theory developed in the following chapters.

2.1 Group rings

Let $\mathbb{Q}$ be a finitely generated abelian group. Structure theorem for finitely generated abelian groups states that $Q \cong \tilde{Q} \times \text{Tor} Q$ where $\tilde{Q}$ is a free abelian group: $\tilde{Q} \cong \mathbb{Z}^n$ for some $n \in \mathbb{Z}_{\geq 0}$, called the rank of $Q$.

Here Tor $Q$ is a finite abelian group, the torsion subgroup of $Q$, consisting of elements in $Q$ of finite order. In fact, most of the results in the essay are independent of Tor $Q$, i.e. most of the results that hold for $\tilde{Q}$ can be shown to hold for $Q$.

Note that this allows us to construct a group homomorphism $\theta : Q \to (\mathbb{R}^n, +)$ such that $\theta(Q) = \mathbb{Z}^n \subseteq \mathbb{R}^n$, $\ker \theta = \text{Tor}(Q)$ and $\theta|_{\tilde{Q}}$ is an isomorphism. We will use symbols $e_1, \ldots, e_n$ to denote the standard basis of $\mathbb{Z}^n$ and symbols $x_1, \ldots, x_n$ for elements of $\tilde{Q}$ with $\theta(x_i) = e_i$.

Now let $R$ be a commutative ring with 1. Then the group ring $RQ$ of $Q$ over $R$ consists of formal linear combinations $a = \sum_{q \in Q} a_q q$, where $a_q \in R \forall q \in Q$ and the set $\text{supp}(a) = \{q \in Q \mid a_q \neq 0\}$, called the support of $a$, is finite. $RQ$ has the structure of a commutative ring with multiplicative identity $1e$ and obvious addition and multiplication.

It is easy to see that the map $x_i \mapsto X_i$ induces an isomorphism from $\tilde{Q}$ to the multiplicative group of Laurent monomials in $n$ variables $X_1, \ldots, X_n$, i.e. the monomials $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, which form a group under multiplication. Thus $R\tilde{Q}$ can be realised as a Laurent polynomial ring over $R$, $R\tilde{Q} \cong R[X^\pm] := R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, consisting of $R$-linear combinations of Laurent monomials, with obvious addition and multiplication. We will always write $f(X) = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha X^\alpha$ for a polynomial $f \in R[X^\pm]$. 
Analogously to the case of $RQ$, we define the support of a polynomial $f \in R[X^\pm]$ to be the set $\text{supp}(f) = \{ \alpha \in \mathbb{Z}^n \mid f_\alpha \neq 0 \}$.

## 2.2 Characters, valuations and weights

As in the previous section, we consider a finitely generated abelian group $Q$. One of the main aspects of the essay concerns characters of $Q$:

**Definition 2.1.** A character of $Q$ is a group homomorphism $\chi : Q \to (\mathbb{R}, +)$.

Note that since $(\mathbb{R}, +)$ is torsion-free, any character of $Q$ will be identically zero on $\text{Tor} Q$. Thus any character $\chi$ of $Q$ is uniquely defined by its restriction $\tilde{\chi} = \chi|_Q$. The group $Q^* = \text{Hom}(Q, \mathbb{R})$ of characters of $Q$ under pointwise addition can be therefore identified with the group $\text{Hom}(\tilde{Q}, \mathbb{R}) \cong \text{Hom}(\mathbb{Z}^n, \mathbb{R})$ of characters of $\tilde{Q}$.

But any character of $\mathbb{Z}^n$ is a function $y \mapsto \langle \chi, y \rangle$ for some $\chi \in \mathbb{R}^n$, and so any character of $Q$ is of the form $q \mapsto \langle \chi, \theta(q) \rangle$ for some $\chi \in \mathbb{R}^n$. This gives a realisation of characters as inner products and shows that $Q^* \cong \mathbb{R}^n$. We will use boldface letters to denote characters in this implementation, i.e. we will write $\langle \chi, \theta(\cdot) \rangle$ for $\chi(\cdot)$.

We may define an equivalence relation $\sim$ on $Q^* \setminus \{0\}$ by positive rescaling: let $\chi \sim \chi'$ if $\chi' = \alpha \chi$ for some $\alpha > 0$. If we denote the equivalence class of $\chi$ by $[\chi]$, we may define the character sphere of $Q$:

$$S(Q) := \{ [\chi] \mid \chi \in Q^* \setminus \{0\} \}.$$  

Then (via isomorphism $Q^* \cong \mathbb{R}^n$) the Euclidean topology on $\mathbb{R}^n$ induces a topology on $S(Q)$ given by the topology on the sphere $S^{n-1}$. An important fact in the following chapters is that $S(Q) \cong S^{n-1}$ is compact with respect to this topology.

Another important concept is given by a valuation of a ring:

**Definition 2.2.** Let $R$ be a commutative ring with 1. We define a valuation on $R$ to be a function $v : R \to \mathbb{R}_{\infty}$ satisfying (for any $r, s \in R$):

(i) $v(r) = \infty$ $\iff$ $r = 0$.

(ii) $v(rs) = v(r) + v(s)$.

(iii) $v(r + s) \geq \min\{v(r), v(s)\}$.

Note that $\{0\} = v^{-1}(\infty)$ is always a prime ideal in $R$ for any valuation $v$. In particular, any ring that has a valuation is an integral domain.

**Remark 2.3.** We claim that, given the axioms (i) and (ii), axiom (iii) is equivalent to

(iii)' $v(r + s) \geq \min\{v(r), v(s)\}$, and equality is attained whenever $v(r) \neq v(s)$.  

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Indeed, clearly (iii)’ ⇒ (iii). To see (iii) ⇒ (iii)’, suppose that \(v(r) \neq v(s)\), w.l.o.g. \(v(r) < v(s)\). Note that (ii) and (i) imply that \(2v(-1) = v(1) = 2v(1)\) and so (as \(v(1) \neq \infty\) by axiom (i)) we have \(v(-1) = v(1) = 0\). Then (iii) and (ii) imply

\[
v(r) \geq \min\{v(-s), v(r + s)\} = \min\{v(-1)v(s), v(r + s)\} = \min\{v(s), v(r + s)\}
\]

and \(v(r) < v(s)\) by the assumption, hence \(v(r) \geq v(r + s)\). It follows that indeed \(v(r + s) = v(r) = \min\{v(r), v(s)\}\), thus the axiom (iii)’ holds.

In particular, if \(r = r_1 + \cdots + r_N\) with \(v(r) > v(r_i)\) \(\forall i\), then an easy inductive argument shows that the minimal value is attained at least twice among the \(v(r_i)\). Hence this must always be the case when \(r = 0\).

In fact, when \(Q\) is torsion-free, any character of \(Q\) together with any valuation on \(R\) induces a valuation on \(R[X^\pm] \cong RQ\):

**Proposition 2.4.** Let \(R\) be an integral domain equipped with a valuation \(v : R \to \mathbb{R}_\infty\), and let \(\chi \in \mathbb{R}^n\) be a character. Define a function

\[
\wt_v : R[X^\pm] \times \mathbb{Z}^n \to \mathbb{R}_\infty,
\]

\[
(f, \alpha) \mapsto v(f\alpha) + \langle \chi, \alpha \rangle,
\]

called \([5]\) the weight with respect to \(\chi\). Then the function

\[
\wt_v : R[X^\pm] \to \mathbb{R}_\infty,
\]

\[
f \mapsto \min\{\wt_v(f, \alpha) \mid \alpha \in \mathbb{Z}^n\},
\]

which is also called the weight with respect to \(\chi\), is a valuation.

**Proof.** First of all, note that for any \(f \in R[X^\pm]\), we have \(v(f\alpha) = \infty\) for all but finitely many \(\alpha \in \mathbb{Z}^n\), hence \(\wt_v : R[X^\pm] \to \mathbb{R}_\infty\) is well-defined. Let \(f, g \in R[X^\pm]\). We check in turn each of the axioms (i), (ii) and (iii) in Definition 2.2:

(i) Clear, since \(v(f\alpha) = \infty \forall \alpha \in \mathbb{Z}^n \Leftrightarrow f = 0\).

(ii) Given any \(\alpha \in \text{supp}(fg)\), let \(s(\alpha) := \{\langle \beta, \gamma \rangle \in \text{supp}(f) \times \text{supp}(g) \mid \beta + \gamma = \alpha\}\); then clearly \(s(\alpha) \neq \emptyset\). Hence, by the definition of a valuation we have

\[
\wt_v(fg, \alpha) = v((fg)\alpha) + \langle \chi, \alpha \rangle \geq \min\{v(f\beta) + v(g\gamma) \mid (\beta, \gamma) \in s(\alpha)\} + \langle \chi, \alpha \rangle
\]

\[
= \min\{v(f\beta) + \langle \chi, \beta \rangle + v(g\gamma) + \langle \chi, \gamma \rangle \mid (\beta, \gamma) \in s(\alpha)\}
\]

\[
\geq \min\{v(f\beta) + \langle \chi, \beta \rangle + v(g\gamma) + \langle \chi, \gamma \rangle \mid \beta \in \text{supp}(f), \gamma \in \text{supp}(g)\}
\]

\[
= \wt_v(f) + \wt_v(g)
\]

and taking the minimum over all \(\alpha \in \text{supp}(fg)\) gives \(\wt_v(fg) \geq \wt_v(f) + \wt_v(g)\).

Conversely, we may define the total order \(<\) to be lexicographic order on \(\mathbb{Z}^n\). Note, in particular, that \(<\) is a term order, i.e. it is invariant under addition of elements of \(\mathbb{Z}^n\): given \(\alpha, \beta, \beta' \in \mathbb{Z}^n\) with \(\beta < \beta'\), we have \(\alpha + \beta < \alpha + \beta'\).
If \( f = 0 \) or \( g = 0 \), then the result is clear, so assume that \( f, g \neq 0 \). Define
\[
m(f) := \min \{ v(f \beta') \mid \beta' \in \mathbb{Z}^n, \, \text{wt}_x(f, \beta') = \text{wt}_x(f) \},
\]
and let
\[
\beta(f) := \min \{ \beta' \in \text{supp}(f) \mid \text{wt}_x(f, \beta') = \text{wt}_x(f), \, v(f \beta') = m(f) \}
\]
where the minimum is taken with respect to lexicographic order. Similarly, define \( m(g) \) and \( \beta(g) \). Let \( \alpha = \beta(f) \beta(g) \) and write
\[
(fg)_\alpha = \sum_{(\gamma(f), \gamma(g)) \in s(\alpha)} f_{\gamma(f)} g_{\gamma(g)}.
\]
We claim that \( v((fg)_\alpha) = v(f_{\beta(f)}) + v(g_{\beta(g)}) \). Indeed, let \( (\gamma(f), \gamma(g)) \in s(\alpha) \) be such that \( v(f_{\gamma(f)} g_{\gamma(g)}) = v(f_{\beta(f)} g_{\beta(g)}) \). Then by the choice of \( \beta(f) \) and \( \beta(g) \) we have
\[
v(f_{\gamma(f)} g_{\gamma(g)}) = v(f_{\beta(f)} g_{\beta(g)}) = v(f_{\beta(f)}) + v(g_{\beta(g)}) = m(f) + m(g)
\]
and so equality must be attained throughout, i.e. we have \( v(f_{\gamma(f)}) = m(f) \) and \( v(g_{\gamma(g)}) = m(g) \). But this implies that
\[
\alpha = \beta(f) + \beta(g) \leq \beta(f) + \gamma(g) \leq \gamma(f) + \gamma(g) = \alpha
\]
by minimality of \( \beta(f) \) and \( \beta(g) \) and since \( \prec \) is a term order, so equality must again be attained throughout. But this implies that \( \gamma(f) = \beta(f) \) and \( \gamma(g) = \beta(g) \), therefore, on the right hand side of \((2.1)\) \( v \) attains its minimal value (among all the terms in the sum) uniquely at \( f_{\beta(f)} g_{\beta(g)} \). Thus axiom (iii)’ in the definition of valuation (see Remark 2.3) implies that \( v((fg)_\alpha) = v(f_{\beta(f)} g_{\beta(g)}) = v(f_{\beta(f)}) + v(g_{\beta(g)}) \), as claimed.

This shows that
\[
\text{wt}_x(fg) \leq \text{wt}_x(fg, \alpha) = v((fg)_\alpha) + \langle x, \alpha \rangle
\]
\[
= v(f_{\beta(f)}) + v(g_{\beta(g)}) + \langle x, \beta(f) \rangle + \langle x, \beta(g) \rangle
\]
\[
= \text{wt}_x(f, \beta(f)) + \text{wt}_x(g, \beta(g)) = \text{wt}_x(f) + \text{wt}_x(g)
\]
and so indeed \( \text{wt}_x(fg) = \text{wt}_x(f) + \text{wt}_x(g) \), as required.

(iii) Given any \( \alpha \in \mathbb{Z}^n \), we have
\[
\text{wt}_x(f + g, \alpha) = v(f + g) + \langle x, \alpha \rangle \geq \min\{ v(f), v(g) \} + \langle x, \alpha \rangle
\]
\[
= \min\{ \text{wt}_x(f, \alpha), \text{wt}_x(g, \alpha) \} \geq \min\{ \text{wt}_x(f), \text{wt}_x(g) \}
\]
and taking the minimum over all \( \alpha \in \mathbb{Z}^n \) gives \( \text{wt}_x(f + g) \geq \min\{ \text{wt}_x(f), \text{wt}_x(g) \} \), as required.

The fact that weights are valuations, especially that they satisfy axiom (ii) of valuations, will turn out to be important. When the valuation \( v : R \to \mathbb{R}_\infty \) is clear, we will write \( \text{wt}_x \) for \( \text{wt}_x^v \).

Valuations and weights will be used in Chapter 3 to define concepts and prove results in tropical geometry. Characters will come in later, in Chapter 4, where finitely generated modules over the ring \( \mathbb{R}Q \) are concerned.
3 Introduction to tropical geometry

Tropical geometry is one of the recent developments of mathematics. In the context of tropical geometry, a totally ordered abelian group (in our case $\mathbb{R}_\infty$) is given the structure of a semiring, where sum and product of two elements is defined to be their minimum and sum, respectively. As in “regular” algebraic geometry, one of the main concerns in tropical geometry is the set of roots of a polynomial, which is defined in tropical setting to be the non-linearity locus of the polynomial [6, Chapter 9].

This chapter introduces the main concepts and results of tropical geometry, which are taken from the book by Diane Maclagan and Bernd Sturmfels [4] unless specified otherwise. These results will prove useful later – in particular, tropical varieties will be shown to be closely related to the geometric invariant defined in Chapter 4.

3.1 Extensions of valuations and tropical varieties

In the context of tropical geometry, we consider Laurent polynomial algebra $K[X^\pm]$ as a special case of the Laurent polynomial ring $R[X^\pm]$, where $K$ is an algebraically closed field. To obtain $K$ from an arbitrary integral domain $R$, we may

(i) let $K' := \text{Frac}(R)$ be the fraction field of $R$, and

(ii) let $K$ be the algebraic closure of $K'$.

It will be important for us to know that every valuation of $R$ can be extended to a valuation of $K$. And indeed, we have

(i) Given valuation $v : R \to \mathbb{R}_\infty$, we may extend it to a valuation $v : K' \to \mathbb{R}_\infty$ by setting

$$v\left(\frac{r}{s}\right) = v(r) - v(s) \quad \text{for } r, s \in R, s \neq 0.$$ 

It is easy to see that $v : K' \to \mathbb{R}_\infty$ is indeed a valuation and that it is the unique valuation extending $v : R \to \mathbb{R}_\infty$.

(ii) Given valuation $v : K' \to \mathbb{R}_\infty$, we may extend it to a valuation $v : K \to \mathbb{R}_\infty$ by setting

$$v(x) = [L : K]^{-1} \times v\left(N_{L/K}(x)\right) = [L : K]^{-1} \times v\left(\prod_{\sigma \in \text{Gal}(L/K)} \sigma(x)\right) \quad \text{for } x \in K,$$
where \( L \) is the splitting field of the minimal polynomial of \( x \). In other words, we define \( v : K \to \mathbb{R}_\infty \) in the only possible way such that it extends \( v : K' \to \mathbb{R}_\infty \), is consistent with axiom (ii) of valuations and that Galois conjugates over \( K' \) are given the same value. Then it is easy to see that \( v : K \to \mathbb{R}_\infty \) satisfies axioms (i) and (ii) of valuations. In fact, it is well-known that it also satisfies axiom (iii) and is indeed a valuation, and it is the unique valuation extending \( v : K' \to \mathbb{R}_\infty \).

This shows that any valuation \( v : R \to \mathbb{R}_\infty \) extends to a unique valuation \( v : K \to \mathbb{R}_\infty \).

In fact, apart from the trivial valuation (defined by \( v(r) = 0 \) whenever \( 0 \neq r \in R \)), we will only consider discrete non-trivial valuations of \( R \), i.e. such that \( v(R) = \mathbb{Z} \cup \{ \infty \} \).

It then follows from the above construction that the value group \( \Gamma \) of \( K \), defined as \( \Gamma := v((K^\times)^n) \) [4, Section 2.1], is a subset of \( \mathbb{Q} \). However, if we let \( p \in R \) be the uniformiser of \( K \) (i.e. any element with \( v(p) = 1 \)), then we have \( v(p^q) = q \) for any \( q \in \mathbb{Q} \), hence the value group is \( \Gamma = \mathbb{Q} \). Note in particular that \( \Gamma \) is dense in \( \mathbb{R} \).

We now fix a valuation \( v : K \to \mathbb{R}_\infty \) in order to define tropical varieties. Consider, as mentioned earlier, the Laurent polynomial algebra in \( n \) variables,

\[
K[X^\pm] = K[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}].
\]

Let \( I \subseteq K[X^\pm] \) be an ideal. Then we may define a set

\[
V(I) := \{ a \in (K^\times)^n \mid f(a) = 0 \ \forall f \in I \},
\]

the variety of \( I \). With this setup we may define the tropical variety of \( I \), and related sets called Bergman complex and Bergman fan of \( I \).

**Definition 3.1.** Let \( I \subseteq K[X^\pm] \) be an ideal. The tropical variety \( \mathcal{T}_v(I) \subseteq \mathbb{R}^n \) of \( I \) associated with the valuation \( v \) is the closure in \( \mathbb{R}^n \) of the set

\[
\mathcal{T}_v'(I) := \{ v(a) \mid a \in V(I) \},
\]

where \( v : (K^\times)^n \to \mathbb{R}^n \) denotes the component-wise valuation, i.e. we write \( v(a_1, \ldots, a_n) \) for \( (v(a_1), \ldots, v(a_n)) \).

Note that \( \mathcal{T}_v'(I) \) is indeed a subset of \( \mathbb{R}^n \) since we have \( V(I) \subseteq (K^\times)^n \), i.e. \( a_i \neq 0 \ \forall i \) whenever \( a = (a_1, \ldots, a_n) \in V(I) \). We write \( V(f) \) for \( V(I) \) and \( \mathcal{T}_v(f) \) for \( \mathcal{T}_v(I) \) if \( I = (f) \) is the principal ideal generated by \( f \in K[X^\pm] \).

A related invariant, describing the behaviour of tropical varieties “at \( \infty \)”, is given by the Bergman complex \( \mathcal{B}_v(I) \) of \( I \): it is the set of points \( y \) in the unit sphere \( S^{n-1} \subseteq \mathbb{R}^n \) defined by:

\[
y \in \mathcal{B}_v(I) \iff \text{there exists a sequence } (y^{(j)})_{j=1}^\infty \text{ in } \mathcal{T}_v'(I) \text{ with } \|y^{(j)}\| \to \infty \text{ and } \frac{y^{(j)}}{\|y^{(j)}\|} \to y \text{ as } j \to \infty.
\]

Bergman complexes were originally introduced as “logarithmic limit sets” by George Bergman in 1971 [2]. A Bergman fan \( \overline{\mathcal{B}}_v(I) \) of \( I \) is the set [6, Chapter 9]

\[
\overline{\mathcal{B}}_v(I) := \{ y \in \mathbb{R}^n \mid y = 0 \text{ or } \frac{y}{\|y\|} \in \mathcal{B}_v(I) \}.
\]
3.2 Initial forms

We now explore some of the geometric properties and alternative definitions of tropical varieties. This section and the following section will provide main results that will be used to prove the relation between valuations and Bieri–Strebel invariants in Chapter 4.

As before, let $K$ be an algebraically closed field and fix a non-trivial valuation $v : K \to \mathbb{R}_\infty$, satisfying $v(p) = 1$ for some $p \in K$, the uniformiser of $K$. We may define the set

$$R_K = \{ a \in K \mid v(a) \geq 0 \},$$

which is a subring of $K$. It is easy to see that $R_K$ is in fact a local ring with unique maximal ideal

$$m_K = \{ a \in K \mid v(a) > 0 \}.$$

We also define

$$k = R_K / m_K,$$

the residue field of $K$ associated with $v$.

Now we claim that $k$ is algebraically closed. Indeed, let $f(X) = X^m + f_{m-1}X^{m-1} + \cdots + f_0$ be a (non-constant) polynomial in $k[X]$. Then $f$ lifts to a polynomial $F(X) = X^m + F_{m-1}X^{m-1} + \cdots + F_0$ in $R_K[X]$ such that $f_i = F_i + m_K \forall i$. Since $K$ is algebraically closed, we may factor $F$ over $K$ as

$$F(X) = (X - a_1) \cdots (X - a_m).$$

Now we have

$$\sum_{i=1}^m v(a_i) = v \left( \prod_{i=1}^m a_i \right) = v((-1)^m F_0) = v(F_0) \geq 0$$

since $F_0 \in R_K$, and so $v(a_i) \geq 0$ for some $i$, i.e. $a_i \in R_K$ for some $i$. It follows that $a_i + m_K \in k$ is a root of $f$. Thus any polynomial over $k$ has a root in $k$, so $k$ is algebraically closed, as claimed.

Fix $\chi \in \mathbb{R}^n$ and recall from Proposition 2.4 that we have a valuation $\text{wt}_\chi = \text{wt}_v^\chi$ of $K[X^\pm]$, the $\chi$-weight. If in addition we have $\chi \in \mathbb{Q}^n$, we may extend this construction further.

Let $\chi = (\chi_1, \ldots, \chi_n) \in \mathbb{Q}^n$ and pick a non-zero polynomial $f \in K[X^\pm]$. Define

$$\tilde{f}(X) = f(p^{\chi_1}X_1, \ldots, p^{\chi_n}X_n) = \sum_{\alpha \in \mathbb{Z}^n} p^{\langle \chi, \alpha \rangle} f_\alpha X^\alpha.$$

Although this depends on the choice of the roots of $p$ whenever $\chi \notin \mathbb{Z}^n$, we may fix a root $p^{1/N}$ for every $N \in \mathbb{Z}_{\geq 1}$ in a consistent way to drop this dependence. Note also that

$$\text{wt}_0 \left( \tilde{f}, \alpha \right) = \text{wt}_\chi(f, \alpha) \forall \alpha \in \mathbb{Z}^n.$$

Define the initial form $\text{in}_\chi^v(f) = \text{in}_\chi(f) \in k[X^\pm]$ of $f$ with respect to $\chi$ to be the reduction modulo $m_K$ of the polynomial $p^{-\text{wt}_\chi(f)} \tilde{f}(X)$; this is well-defined since (by the assumption)
the value group of $v$ is $\Gamma = \mathbb{Q}$, hence $\text{wt}_\chi(f) \in \mathbb{Q}$. We can see that $\text{in}_\chi(f)$ has non-zero coefficients at precisely those $\alpha \in \mathbb{Z}^n$ that correspond to the terms of $f$ of minimal $\chi$-weight. We also define $\text{in}_\chi(0) = 0$.

Similarly, for an ideal $I \subseteq K[\mathbb{X}^\pm]$, we define the initial form of $I$,

$$\text{in}_\chi^*(I) = \text{in}_\chi(I) := \{ \text{in}_\chi(f) \mid f \in I \}.$$  

Then $\text{in}_\chi(I)$ is itself an ideal:

**Lemma 3.2.** $\text{in}_\chi(I)$ is an ideal in $k[\mathbb{X}^\pm]$ for any $I \subseteq K[\mathbb{X}^\pm]$ and any $\chi \in \mathbb{Q}^n$.

**Proof.** It is easy to see that $\text{in}_\chi(f) + \text{in}_\chi(g) = \text{in}_\chi(p^{-\text{wt}_\chi(f)} f + p^{-\text{wt}_\chi(g)} g)$ for any $f, g \in I$, so $\text{in}_\chi(I)$ is closed under addition. Hence we only need to show that $\text{in}_\chi(I)$ is invariant under multiplication by elements of $k[\mathbb{X}^\pm]$.

Clearly any element of $g' \in k[\mathbb{X}^\pm]$ can be lifted to an element $g \in R_K[\mathbb{X}^\pm]$ such that $g' = \text{in}_\chi(g)$. Thus it would be enough to show that

$$\text{in}_\chi(f) \text{in}_\chi(g) = \text{in}_\chi(fg)$$

for any $f, g \in K[\mathbb{X}^\pm]$. But $\text{wt}_\chi$ is a valuation by Proposition 2.4, so terms of minimal $\chi$-weight do not “cancel” when polynomials are multiplied, i.e. $\text{wt}_\chi(f) + \text{wt}_\chi(g) = \text{wt}_\chi(fg)$.

It follows that

$$p^{-\text{wt}_\chi(f)}\tilde{f}(\mathbf{X}) \times p^{-\text{wt}_\chi(g)}\tilde{g}(\mathbf{X}) = \sum_{\alpha \in \mathbb{Z}^n} \sum_{\beta \in \mathbb{Z}^n} p^{(\chi, \alpha) + (\chi, \beta) - \text{wt}_\chi(f) - \text{wt}_\chi(g)} f_{\alpha}g_{\beta}X^{\alpha + \beta}$$

$$= \sum_{\gamma \in \mathbb{Z}^n} p^{(\chi, \gamma) - \text{wt}_\chi(fg)} (fg)_{\gamma}X^{\gamma} = p^{-\text{wt}_\chi(fg)} (\tilde{fg})(\mathbf{X}).$$

Taking reduction modulo $\mathfrak{m}_K$ of the equality above, it follows that $\text{in}_\chi(f) \text{in}_\chi(g) = \text{in}_\chi(fg)$, as required.

Finally, we define a tropical basis for $I$ with respect to $v$ to be a finite generating set $\mathcal{G}$ of $I$ such that for each $\chi \in \mathbb{Q}^n$ with $\text{in}_\chi(I) = K[\mathbb{X}^\pm]$, the set $\{ \text{in}_\chi(g) \mid g \in \mathcal{G} \}$ contains a monomial. As the units in $K[\mathbb{X}^\pm]$ are precisely the monomials, this is equivalent to saying that $\text{in}_\chi(I)$ contains a monomial if and only if $\{ \text{in}_\chi(g) \mid g \in \mathcal{G} \}$ does. We will use the following result [4, Theorem 2.6.5], which we do not prove here:

**Proposition 3.3.** Every ideal $I \subseteq K[\mathbb{X}^\pm]$ has a (finite) tropical basis.

### 3.3 Geometry of tropical varieties and Bergman fans

We now return to tropical varieties. Throughout this section, we assume that the value group of the (fixed) valuation $v$ is $\Gamma = \mathbb{Q}$.

We first consider tropical varieties of principal ideals. Let $f \in K[\mathbb{X}^\pm]$ be a non-zero Laurent polynomial. Define $\mathcal{S}_\chi(f)$ to be the set of such $\chi \in \mathbb{R}^n$ that the minimal $\chi$-weight among the terms of $f$ is attained at least twice.
Note that by definition
\[ S_v(f) = \bigcup_{\alpha \neq \beta \in \text{supp}(f)} \{ \chi \in \mathbb{R}^n \mid \text{wt}_\chi(f, \alpha) = \text{wt}_\chi(f, \beta) \geq \text{wt}_\chi(f, \gamma) \forall \gamma \in \mathbb{Z}^n \} \]
\[ = \bigcup_{\alpha \neq \beta \in \text{supp}(f)} H_0(\beta - \alpha, v(f_\alpha/f_\beta)) \cap \cap_{\gamma \in \text{supp}(f)} H_+(\gamma - \alpha, v(f_\alpha/f_\gamma)) \]
where \( H_+(\gamma, c) \) is the closed halfspace \( \{ \chi \in \mathbb{R}^n \mid \langle \chi, \gamma \rangle \geq c \} \), and \( H_0(\gamma, c) \) is the hyperplane \( \{ \chi \in \mathbb{R}^n \mid \langle \chi, \gamma \rangle = c \} = H_+(\gamma, c) \cap H_+(-\gamma, -c) \). Thus \( S_v(f) \) is a polyhedral complex, i.e. a finite union of convex polyhedra; here we define a convex polyhedron to be a (not necessarily bounded) intersection of finitely many closed halfspaces. Moreover, as by assumption the value group of \( v \) is \( \Gamma = \mathbb{Q} \), this polyhedral complex is rational, i.e. all the closed halfspaces \( H_+(\gamma, c) \) involved have \( \gamma \in \mathbb{Q}^n \) and \( c \in \mathbb{Q} \).

In fact, this polyhedral complex is precisely the tropical variety of the principal ideal \( (f) \):

**Lemma 3.4.** \( S_v(f) = \mathcal{T}_v(f) \).

**Proof.** We will show inclusions (\( \supseteq \)) and (\( \subseteq \)) in the statement.

(\( \supseteq \)) Since \( S_v(f) \) is closed in \( \mathbb{R}^n \) and \( \mathcal{T}_v(f) \) is by definition the closure of \( \mathcal{T}'_v(f) \), it is enough to show that \( S_v(f) \supseteq \mathcal{T}'_v(f) \). Thus, let \( \chi \in \mathcal{T}'_v(f) \): then \( \chi = v(a) \) for some \( a \in V(f) \). In particular, we have
\[ \infty = v(0) = v(f(a)) \geq \min\{ v(f_\alpha a^\alpha) \mid \alpha \in \mathbb{Z}^n \} = \min\{ \text{wt}_v(a)(f, \alpha) \mid \alpha \in \mathbb{Z}^n \} \]
and the inequality is strict since \( f \neq 0 \), so the minimum must be attained at least twice on the right hand side. Thus \( \chi = v(a) \in S_v(f) \), as required.

(\( \subseteq \)) Since \( S_v(f) \) is a rational polyhedral complex, it is easy to see that the points in \( \mathbb{Q}^n \) form a dense subset of \( S_v(f) \). Thus it is enough to show that \( S_v(f) \cap \mathbb{Q}^n \subseteq \mathcal{T}_v(f) \). Therefore, let \( \chi = (\chi_1, \ldots, \chi_n) \in S_v(f) \cap \mathbb{Q}^n \), i.e. let \( \chi \in \mathbb{Q}^n \) be such that
\[ c := \text{wt}_\chi(f, \alpha) = \text{wt}_\chi(f, \beta) \leq \text{wt}_\chi(f, \gamma) \quad \forall \gamma \in \mathbb{Z}^n \]
for some \( \alpha, \beta \in \text{supp}(f) \) with \( \alpha \neq \beta \); w.l.o.g. we may assume that \( \alpha_n \neq \beta_n \).

Choose any \( a_1, \ldots, a_{n-1} \in K^\times \) such that \( \chi_i = v(a_i) \) for \( i = 1, \ldots, n-1 \). Define \( \varphi(X) = \sum_{\xi \in \mathcal{Z}} \varphi_\xi X^\xi = f(a_1, \ldots, a_{n-1}, X) \in K[X, X^{-1}] \), and consider the Newton polygon \( \mathcal{N}(\varphi) \) of \( \varphi \). Note that for each \( \zeta \in \mathbb{Z} \) we have
\[ v(\varphi_\zeta) \geq \min\{ v(f_\gamma a_1^{\gamma_1} \cdots a_{n-1}^{\gamma_{n-1}}) \mid \gamma_n = \zeta \} \]
\[ = \min\{ v(f_\gamma) + \langle \chi', \gamma' \rangle \mid \gamma_n = \zeta \} \quad (3.1) \]
where given an \( n \)-tuple \( y = (y_1, \ldots, y_n) \) we write \( y' \) for \( (y_1, \ldots, y_{n-1}) \).

We claim that there exists a suitable choice of \( \alpha' = (a_1, \ldots, a_{n-1}) \) such that the inequality in (3.1) becomes an equality for both \( \zeta = \alpha_n \) and \( \zeta = \beta_n \). Indeed, this is clear for \( n = 1 \). If \( n \geq 2 \), we define \( f^{\zeta} \in K[X_1, \ldots, X_{n-1}, X_1^{-1}, \ldots, X_{n-1}^{-1}] \) by
\[ f^{\zeta}(X_1, \ldots, X_{n-1}) = \sum_{\gamma \in \mathbb{Z}^{n-1}} f_\gamma X_1^{\gamma_1} \cdots X_{n-1}^{\gamma_{n-1}} \]
for any $\zeta \in \mathbb{Z}$, and we consider variety $V_\zeta := V(\text{in}_\chi(f^\zeta)) \subseteq (k^\times)^{n-1}$. Then the claim is equivalent to saying that $V_{\alpha_n} \cup V_{\beta_n}$ is a proper subset of $(k^\times)^{n-1}$. However, we have $V_{\alpha_n} \cup V_{\beta_n} = V(\text{in}_\chi(f^{\alpha_n}) \text{in}_\chi(f^{\beta_n}))$, which is a proper subset of $(k^\times)^{n-1}$ since $k$ is algebraically closed and both polynomials are non-zero. This proves the claim.

Now consider the line $\ell$ on $\mathbb{R}^2$ defined by the equation $Y = c - y_n X$. By the previous paragraph, the points $(\alpha_n, v(\varphi_{\alpha_n}))$ and $(\beta_n, v(\varphi_{\beta_n}))$ lie on $\ell$; moreover, the point $(\zeta, v(\varphi_{\zeta}))$ does not lie below $\ell$ for any $\zeta \in \mathbb{Z}$. It follows that an edge of $N(\varphi)$ lies in $\ell$, so in particular an edge of $N(\varphi)$ has slope $-y_n$. Thus $\varphi$ has a root $a_n$ such that $y_n = v(a_n)$, and so $f$ has a root $a$ such that $\chi = v(a)$. Hence $\chi \in \mathcal{T}_v(f) \subseteq \mathcal{T}_v(f)$, as required.

We would like to know how the tropical variety $\mathcal{T}_v(I)$ of an ideal $I \subseteq K[\mathbf{X}^\pm]$ is related to the varieties $\mathcal{T}_v(f)$ for elements $f \in I$. One relation is clear: if $\chi \in \mathcal{T}_v(I)$, then there exists a sequence $(a^{(j)})$ of points in $V_1$ such that $v(a^{(j)}) \to \chi$ as $j \to \infty$; but then, given any $f \in I$, we also have $a^{(j)} \in V(f)$ for each $j$, so $\chi \in \mathcal{T}_v(f)$. It follows that $\mathcal{T}_v(I) \subseteq \bigcap_{f \in I} \mathcal{T}_v(f)$.

We claim that this inclusion is in fact an equality. To prove this we use the following Lemma:

**Lemma 3.5.** The intersection $\bigcap_{f \in I} \mathcal{T}_v(f)$ is a finite intersection – in particular,

$$\bigcap_{f \in G} \mathcal{T}_v(f) = \bigcap_{f \in I} \mathcal{T}_v(f),$$

where the finite subset $G \subseteq I$ is a tropical basis for $I$.

**Proof.** We clearly have $\bigcap_{f \in G} \mathcal{T}_v(f) \supseteq \bigcap_{f \in I} \mathcal{T}_v(f)$ for any subset $G \subseteq I$. Conversely, recall that if $G \subseteq I$ is a (finite) tropical basis for $I$ (which exists by Proposition 3.3), then for any $\chi \in \mathbb{Q}^n$ such that $\text{in}_\chi(I)$ contains a monomial, the set $\{\text{in}_\chi(f) \mid f \in G\}$ also contains a monomial. Combining this with Lemma 3.4 shows that

$$\mathbb{Q}^n \cap \bigcap_{f \in G} \mathcal{T}_v(f) = \mathbb{Q}^n \cap \bigcap_{f \in I} \mathcal{S}_v(f) = \mathbb{Q}^n \cap \bigcap_{f \in G} \mathcal{S}_v(f) = \mathbb{Q}^n \cap \bigcap_{f \in I} \mathcal{T}_v(f). \tag{3.2}$$

Now the intersection $\bigcap_{f \in G} \mathcal{S}_v(f) = \bigcap_{f \in G} \mathcal{T}_v(f)$ is a finite intersection of rational polyhedral complexes, so is itself a rational polyhedral complex – in particular, the points of $\mathbb{Q}^n$ are dense in this set. But $\bigcap_{f \in I} \mathcal{T}_v(f)$ is closed (as an intersection of closed sets), hence (3.2) shows that indeed $\bigcap_{f \in G} \mathcal{T}_v(f) \subseteq \bigcap_{f \in I} \mathcal{T}_v(f)$, as required. 

Note that Lemma 3.5 and Lemma 3.4 combine to give

$$\bigcap_{f \in G} \mathcal{S}_v(f) = \bigcap_{f \in I} \mathcal{S}_v(f).$$

This gives a new definition of tropical bases, which provides information about all characters and not just the rational ones:
Corollary 3.6. Let $\chi \in \mathbb{R}^n$ and suppose that there exists $f \in I$ with the following property:

$$\text{for some } \alpha \in \text{supp}(f), \text{ we have } \text{wt}^\chi(f, \alpha) < \text{wt}^\chi(f, \beta) \ \forall \beta \in \text{supp}(f) \setminus \{\alpha\}.$$  

Then there exists $f \in \mathcal{G}$ with the same property. \hfill \square

Now Lemma 3.5 implies that the set $\bigcap_{f \in I} T_v(f) = \bigcap_{f \in I} S_v(f)$ is a rational polyhedral complex, so the points of $\mathbb{Q}^n$ are dense in this set. So it is sufficient to show that $T_v(I) \supseteq \mathbb{Q}^n \cap \bigcap_{f \in I} S_v(f)$.

In other words, it is enough to show the following:

For any $\chi \in \mathbb{Q}^n$ such that $\text{in}^\chi(I)$ contains no monomials, we have $\chi \in T_v(I)$. (*)

It is a fact, which is not proved here, that the following holds for prime ideals [4, Proposition 3.2.11]:

**Theorem 3.7.** Let $\mathfrak{P} \leq K[X^\pm]$ be a prime ideal and let $v : K \to \mathbb{R}_\infty$ be a valuation. Let $\chi \in \Gamma^n$ be such that $\text{in}^\chi(\mathfrak{P})$ contains no monomials, and let $a \in V(\text{in}^v(\mathfrak{P})) \subseteq (k^\times)^n$ (which exists since $\text{in}^\chi(\mathfrak{P})$ is a proper ideal and $k$ is algebraically closed). Then a “lifts” to a point in $V(\mathfrak{P}) \subseteq (K^\times)^n$, i.e. there exists $b \in V(\mathfrak{P})$ such that $v(b) = \chi$ and $a = p^{-\chi}b + m_K$.

Given Theorem 3.7, we may finalise the proof of (*). Indeed, since $\text{in}^\chi(f)^m = \text{in}^\chi(f^m)$ for any $f \in K[X^\pm]$ and any $m \in \mathbb{Z}_{\geq 1}$ (by the proof of Lemma 3.2), we have $\text{in}^\chi(I) = \text{in}^\chi(\sqrt{I})$. As clearly $V(I) = V(\sqrt{I})$ and so $T_v(I) = T_v(\sqrt{I})$, we may w.l.o.g. assume that $I$ is radical.

Thus $V(I) = V(\mathfrak{P}_1) \cup \cdots \cup V(\mathfrak{P}_r)$ for some prime ideals $\mathfrak{P}_i \leq K[X^\pm]$, for which we have $I = \bigcap_{i=1}^r \mathfrak{P}_i$ by the Nullstellensatz. But then, for some $i$, $\text{in}^\chi(\mathfrak{P}_i)$ does not contain a monomial: indeed, otherwise for each $i$ we could pick $f_i \in \mathfrak{P}_i$, with $\text{in}^\chi(f_i) = 1$, in which case $f := \prod_{i=1}^r f_i \in I$ would satisfy $\text{in}^\chi(f) = 1$, contradicting our choice of $\chi$. Thus the Theorem implies that $\chi \in T_v(I)$, which concludes the proof of (*), and therefore the claim. Hence, we have proved:

**Theorem 3.8.** Let $I \leq K[X^\pm]$ be an ideal. Then

$$T_v(I) = \bigcap_{f \in I} T_v(f).$$ \hfill \square

In fact, Theorem 3.8 is the crucial part of one of the main results in tropical geometry, known as the Fundamental Theorem of Tropical Algebraic Geometry [4, Theorem 3.2.5]. Combining Theorem 3.8, Lemma 3.5 and Lemma 3.4 also gives an important geometric result:

**Corollary 3.9.** $T_v(I)$ is a rational polyhedral complex. \hfill \square
Now recall the Bergman fan $\tilde{B}_v(I)$ of $I$. Since $T_v(I)$ is a polyhedral complex, it is easy to see that $\tilde{B}_v(I)$ is precisely the set of such $\chi \in \mathbb{R}^n$ that $T_v(I)$ contains a ray of slope $\chi$, i.e.

$$\{y + c\chi \mid c \in [0, \infty)\} \subseteq T_v(I) \quad \text{for some } y \in \mathbb{R}^n.$$ 

In particular, $\tilde{B}_v(I)$ is the set of such $\chi \in \mathbb{R}^n$ that for all $g \in \mathcal{G}$, we have $\{y_g + c\chi \mid c \in [0, \infty)\} \subseteq T_v(g)$ for some $y_g \in \mathbb{R}^n$ (where, as before, $\mathcal{G}$ is the tropical basis for $I$).

Lemma 3.4 gives an explicit description of polyhedral complexes $T_v(g)$, which implies the following:

**Corollary 3.10.**

$$\tilde{B}_v(I) = \bigcap_{f \in \mathcal{G}} \{\chi \in \mathbb{R}^n \mid \langle \chi, \alpha \rangle = \langle \chi, f \rangle \text{ for at least 2 values of } \alpha \in \text{supp}(f)\} = \bigcap_{f \in \mathcal{G}} S_0(f).$$

This concludes the chapter on tropical geometry. The essay now turns to analysis of a different concept – Bieri–Strebel invariants of a module – whose introduction was motivated by group-theoretic problems.
4 Bieri–Strebel invariants

We now consider another construction, which was done by Robert Bieri and Ralph Strebel in 1980s [3] (on which first two sections of this chapter are based on), independently of the work done by George Bergman in 1970s [2]. The motivation behind this construction was primarily group theoretic, and is discussed in Chapter 5.

For the first two sections of this chapter, we let \( Q \) be a finitely generated abelian group, but we do not require it to be torsion-free. We let \( A \) be a finitely generated left \( R^Q \)-module. We will introduce a subset of \( S(Q) \cong S^{n-1} \) depending on \( A \), and will show that this subset is closely related to tropical varieties of a certain ideal.

4.1 Definitions

Let \( \chi : Q \rightarrow \mathbb{R} \) be a non-zero character of the finitely generated abelian group \( Q \). We may define a subset of \( Q \) by

\[
Q_{\chi} = \{ q \in Q \mid \chi(q) \geq 0 \}.
\]

Since \( \chi \) is a group homomorphism, this subset is closed under multiplication and contains the identity (as \( \chi(e) = 0 \)), so in fact \( Q_{\chi} \) is a monoid. This allows us to define the monoid ring \( R^Q_{\chi} \) the same way the group ring \( R^Q \) was defined. Note that clearly \( Q_{\chi} = Q_{\alpha \chi} \) for any \( \alpha > 0 \), so for a non-zero \( \chi \), \( Q_{\chi} \) depends only on the equivalence class of \( \chi \) in \( S(Q) \).

Now we let \( A \) be a left \( R^Q \)-module that is finitely generated over \( R^Q \). Then \( A \) may or may not be finitely generated over the subring \( R^Q_{\chi} \subseteq R^Q \). This allows us to introduce the following:

**Definition 4.1.** A Bieri–Strebel invariant for the module \( A \) is the subset

\[
\Sigma_A = \Sigma_A(Q) = \{ [\chi] \in S(Q) \mid A \text{ is finitely generated over } R^Q_{\chi} \}.
\]

Recall the weight function, introduced in the case when \( Q \) is torsion-free in proposition 2.4, whose definition can be extended in our case to the function \( \text{wt}_{\chi} \) defined by

\[
\text{wt}_{\chi}^v(f, q) = v(f_q) + \chi(q) \quad \text{for } f = \sum_{q \in Q} f_q q \in R^Q \text{ and } q \in Q
\]

and

\[
\text{wt}_{\chi}^c(f) = \min \{ \text{wt}_{\chi}(f, q) \mid q \in Q \} \quad \text{for } f \in R^Q.
\]
For this section and the following section, we take the valuation $v$ to be trivial (i.e. $v = 0$ on $R \setminus \{0\}$), and we write $w_\chi$ for $w_\chi^0$. Note that although $w_\chi$ is not a valuation when $Q$ is not torsion-free (as axiom (ii) of valuations does not hold), proof of Proposition 2.4 shows that we still have

$$w_\chi(fg) \geq w_\chi(f) + w_\chi(g) \quad \forall f, g \in RQ.$$ 

The following proposition will provide a useful alternative definition of $\Sigma_A$.

**Proposition 4.2.** Let $C(A) = C_{RQ}(A)$ be the centraliser of $A$ in $RQ$, and let $\chi: Q \to \mathbb{R}$ be a non-zero character. Then

$$[\chi] \in \Sigma_A \iff w_\chi(f) > 0 \text{ for some } f \in C(A).$$

This shows in particular that

$$\Sigma_A = \bigcup_{f \in C(A)} \{[\chi] \in S(Q) \mid w_\chi(f) > 0\}.$$ 

**Proof.** ($\Leftarrow$) Suppose $w_\chi(f) > 0$ for some $f \in C(A)$. Let $\mathcal{A}$ be a finite generating set for $A$ over $RQ$.

Let $g \in RQ$ and $a \in \mathcal{A}$ be arbitrary. Then $ga = gf^m a$ for any $m \in \mathbb{Z}$, but $w_\chi(gf^m) \geq w_\chi(g) + m w_\chi(f) \geq 0$ whenever $m \geq -w_\chi(g)/w_\chi(f)$. In particular, we have

$$ga = gf^m a \in RQ_\chi A.$$ 

Taking a linear span over all $g \in RQ$ and all $a \in \mathcal{A}$ yields

$$A = RQA \subseteq RQ_\chi A,$$

thus $RQ_\chi A = A$, i.e. $\mathcal{A}$ is also a finite generating set for $A$ over $RQ_\chi$, as required.

($\Rightarrow$) Suppose that $\mathcal{A} = \{a_1, \ldots, a_r\}$ is a (finite) generating set for $A$ over $RQ_\chi$. This means that any $a \in A$ can be written as a $RQ_\chi$-linear combination of the $a_i$. In particular, choose any $q \in Q$ with $\chi(q) < 0$ (this is possible since $\chi$ is a non-zero homomorphism). Then we can write

$$qa_i = \sum_{j=1}^r f_{ij} a_j \quad \text{for } i = 1, \ldots, r$$

where $f_{ij} \in RQ_\chi$, or equivalently (after rearranging),

$$\sum_{j=1}^r (\delta_{ij} - q^{-1} f_{ij}) a_j = 0 \quad \text{for } i = 1, \ldots, r.$$ 

As $RQ$ is commutative, this implies that the element

$$g := \prod_{i=1}^r \sum_{j=1}^r (\delta_{ij} - q^{-1} f_{ij}) \in RQ$$

(4.1)
annihilates \(A\), so \(g\) also annihilates \(A\) by extending linearly. In particular, \(1 - g \in C(A)\).

But since \(q^{-1} \in Q\chi\) by the choice of \(\chi\), by expanding (4.1) we can see that \(g = 1 - q^{-1}h\) for some \(h \in RQ\chi\), so in particular \(q^{-1}h = 1 - g \in C(A)\) and \(\text{wt}_\chi(q^{-1}h) \geq -\chi(q) + \text{wt}_\chi(h) \geq -\chi(q) > 0\). Thus \(q^{-1}h\) is the required element of \(C(A)\).

Recall that \(Q \cong \tilde{Q} \times \text{Tor}(Q)\) for a torsion-free finitely generated abelian group \(\tilde{Q} \cong \mathbb{Z}^n\). Proposition 4.2 immediately implies that

\[
\Sigma_A(Q) = \Sigma_{\tilde{A}}(\tilde{Q})
\]

as any character of \(Q\) is identically zero on \(\text{Tor}\ Q\). Since \(\text{Tor}\ Q \cong Q/\tilde{Q}\) is finite, this is also easily seen directly from Definition 4.1: indeed, if \(Q\) is a (finite) transversal of \(\tilde{Q}\) in \(Q\) and \(A\) is a finite generating set of \(A\) over \(RQ\), then \(QA = \{qa \mid q \in Q, a \in A\}\) is a finite generating set of \(A\) over \(\tilde{Q}\).

Moreover, \(\Sigma_A\) only depends on \(\text{Ann}(A) = \text{Ann}_{RQ}(A) := C(A) - 1\), the annihilator ideal of \(A\) in \(RQ\). But if we view \(A' = RQ/\text{Ann}(A)\) as a left \(RQ\)-module with the action given by left multiplication, it is clear that \(\text{Ann}(A') = \text{Ann}(A)\). Since \(A'\) is finitely generated over \(RQ\) (e.g. by \(1e + \text{Ann}(A) \in A'\)), this implies that

\[
\Sigma_A = \Sigma_{RQ/\text{Ann}(A)}.
\]

### 4.2 Conditions on finite generation of \(A\)

This section examines what happens when the \(RQ\)-module \(A\) is regarded as an \(R\)-module. In particular, as \(R = Re\) is a subring of \(RQ\chi\) for any character \(\chi : Q \to \mathbb{R}\), it follows that if \(A\) is finitely generated over \(R\), then it is also finitely generated over each \(RQ\chi\). It turns out that this necessary condition is also sufficient, and the aim of this section is to prove this.

We now introduce a geometric lemma that we will use in the proofs. Recall, for a finite subset \(F \subseteq \mathbb{R}^n\) and an element \(\chi \in \mathbb{R}^n\) we write \(\langle \chi, F \rangle\) for \(\min\{\langle \chi, y \rangle \mid y \in F\}\). Now suppose that we have a finite subset \(F \subseteq \mathbb{R}^n\) such that there exists some \(\chi \in \mathbb{R}^n\) with \(\langle \chi, F \rangle > 0\), and w.l.o.g. \(\|\chi\| = 1\). Since \(F\) is finite, it is easy to see that with \(\rho\) large enough, \(F\) is contained in an open ball of radius \(\rho\) whose boundary goes through the origin and has inward normal \(\chi\) there. That is, \(F\) is contained in the open ball \(\rho\chi + B_\rho\) (where \(B_\rho = \{y \in \mathbb{R}^n \mid \|y\| < \rho\}\)) for \(\rho\) large enough: see the picture below.
Moreover, by finiteness of $F$ and openness of $\rho \chi + B_\rho$, $F$ is still contained in this ball if we translate it some small distance – in particular, $F$ is contained in the ball $(\rho + \varepsilon) \chi + B_\rho$ for some $\varepsilon > 0$. The following lemma says that this property is in a sense universal:

**Lemma 4.3.** Let $\mathcal{F}$ be a finite collection of finite subsets of $\mathbb{R}^n$, and suppose that for each $\chi \in \mathbb{R}^n \setminus \{0\}$ there exists an $F \in \mathcal{F}$ with $\langle \chi, F \rangle > 0$. Then there exist $\rho_0 > 0$ and $\varepsilon > 0$ such that for each $\rho > \rho_0$ and for each $\chi \in B_{\rho + \varepsilon}$, there exists $F \in \mathcal{F}$ with $F \subseteq \chi + B_\rho$.

**Proof.** Define a function $E : S^{n-1} \to \mathbb{R}$ from the unit sphere in $\mathbb{R}^n$ to the reals by

$$E(\chi) := \max \{ \langle \chi, F \rangle \mid F \in \mathcal{F} \}.$$

By the assumption, $E(\chi) > 0 \ \forall \chi \in S^{n-1}$. But $S^{n-1}$ is compact and $E$ is clearly continuous, hence there exists $\chi_0 \in S^{n-1}$ such that

$$E(\chi_0) = E_0 := \inf \{ E(\chi) \mid \chi \in S^{n-1} \}.$$

In particular, $E_0 > 0$.

Now let $\varepsilon = E_0/2$. Let $\chi \in S^{n-1}$ and pick $F = F(\chi) \in \mathcal{F}$ with $\langle \chi, F \rangle \geq E_0 = 2\varepsilon$. Then with $\rho > 0$ sufficiently large, $F$ is contained in both $B_\rho$ and $(\rho + \varepsilon) \chi + B_\rho$, i.e. $F$ is contained in the purple region in the image below.
In particular, the following construction is possible:

- Let \( \rho'_1 = \max \{ \| y \| \mid y \in F \} \), so that \( F \in B_\rho \forall \rho > \rho_1 \).

- Let \( \rho''_1 = \max \left\{ \frac{\| y \|^2 - 2 \langle \chi, y \rangle \varepsilon + \varepsilon^2}{2(\langle \chi, y \rangle - \varepsilon)} \mid y \in F \right\} \). Note that given \( \rho > \rho''_1 \) and \( y \in F \) we have
  \[
  2 \left( \langle \chi, y \rangle - \varepsilon \right) \rho > \| y \|^2 - 2 \langle \chi, y \rangle \varepsilon + \varepsilon^2 = \left( \langle \chi, y \rangle - \varepsilon \right)^2 + \| y - \langle \chi, y \rangle \chi \|^2
  \]
  which gives (after rearranging)
  \[
  (\rho + \varepsilon - \langle \chi, y \rangle)^2 + \| y - \langle \chi, y \rangle \chi \|^2 < \rho''_1,
  \]
  i.e. \( y \in (\rho + \varepsilon) \chi + B_\rho \).

- Let \( \rho_1 = \max \{ \rho'_1, \rho''_1 \} \): then
  \[
  F \subseteq B_\rho \cap [(\rho + \varepsilon) \chi + B_\rho]
  \]
  for any \( \rho > \rho_1 \). Since \( B_\rho \) is convex, this also implies that \( F \subseteq c \chi + B_\rho \) for any \( c \in [0, \rho + \varepsilon) \).

Now for each \( F \in \mathcal{F} \), the set \( A_F := \{ \chi \in S^{n-1} \mid \langle \chi, F \rangle \geq E_0 \} \) is a closed subset of a compact topological space \( S^{n-1} \), therefore is compact. On each \( A_F \), using the construction above, \( \rho_1 \) is defined and varies continuously with \( \chi \), therefore it is bounded above (by \( \rho_F \), say) using compactness of \( A_F \).

Finally, let \( \rho_0 = \max \{ \rho_F \mid F \in \mathcal{F} \} \). Then \( \rho_1 \leq \rho_0 \) whenever \( \rho_1 \) is defined, and so for each \( \chi \in S^{n-1} \), there exists an \( F \in \mathcal{F} \) with \( F \subseteq c \chi + B_\rho \) for any \( c \in [0, \rho + \varepsilon) \) and any \( \rho > \rho_0 \). It follows that for any \( \chi \in B_{\rho + \varepsilon} \) and any \( \rho > \rho_0 \), we have \( F \subseteq \chi + B_\rho \) for some \( F \in \mathcal{F} \), as required.

We can use this Lemma to prove the main result of this section – the condition for the left \( RQ \)-module \( A \) to be finitely generated as an \( R \)-module. Namely,

**Theorem 4.4.** Let \( Q \) be a finitely generated abelian group and let \( A \) be a finitely generated left \( RQ \)-module. Then

\[
A \text{ is finitely generated over } R \iff \Sigma_A = S(Q).
\]

**Proof.** (\( \Rightarrow \)) Any set that generates \( A \) over \( R \) also generates \( A \) over \( RQ_\chi \) for any character \( \chi \), hence if \( A \) is finitely generated over \( R \) then it is finitely generated over each \( RQ_\chi \). Thus \( \Sigma_A = S(Q) \).

(\( \Leftarrow \)) If \( \Sigma_A = S(Q) \), then Proposition 4.2 implies that

\[
S(Q) = \bigcup_{f \in C(A)} \{ [\chi] \in S(Q) \mid \text{wt}_\chi(f) > 0 \}
\]

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is an open cover of a compact topological space $S(Q)$. Thus there exists a finite subcover, i.e. there exists a finite subset $\mathcal{F}' \subseteq C(A)$ such that

$$S(Q) = \bigcup_{f \in \mathcal{F}'} \{[\chi] \in S(Q) \mid \text{wt}_\chi(f) > 0\}.$$ 

Recall that we have a group homomorphism $\theta : Q \to (\mathbb{R}^n, +)$ with finite kernel $\text{Tor}(Q)$, such that $\theta$ induces an isomorphism $Q / \text{Tor}(Q) \cong \mathbb{R}^n$. In particular, for each $\rho > 0$, the preimage $C_\rho := \theta^{-1}(B_\rho) \subseteq Q$ of the open ball $B_\rho$ is finite. Moreover, any $q \in Q$ has $||\theta(q)||^2 \in R$, hence $C_\rho$ is constant when $\rho \in (\sqrt{k}, \sqrt{k + 1}]$ for any $k \in R$.

In order to apply Lemma 4.3, the subset $\mathcal{F}' \subseteq C(A)$ can be mapped to a collection of finite subsets of $\mathbb{R}^n$ by setting

$$\mathcal{F} := \{\theta(\text{supp}(f)) \mid f \in \mathcal{F}'\}.$$ 

We can now apply Lemma 4.3 to the collection $\mathcal{F}$ of subsets of $\mathbb{R}^n$. In particular, Lemma 4.3 implies that there exist $\rho_0 > 0$ and $\varepsilon > 0$ such that for each $\rho > \rho_0$ and for each $\alpha \in B_{\rho+\varepsilon}$, there exists $f \in \mathcal{F}'$ with $\alpha + \theta(\text{supp}(f)) \subseteq B_\rho$. Thus if we take any $q \in C_{\rho+\varepsilon}$ then $\theta(q) \in B_{\rho}$ and the Lemma implies that $\text{supp}(qf) = q \text{supp}(f) \subseteq C_\rho$ for some $f \in \mathcal{F}'$.

Now recall that $C_\rho = C_{\sqrt{k+1}}$ for each $k \in R$ and $\rho \in (\sqrt{k}, \sqrt{k + 1}]$ – in particular,

$$C_{\sqrt{k+\varepsilon}} \supseteq C_{\sqrt{k+1}} \quad \forall k \in R_{\geq 0}.$$ 

Now fix an integer $k > \rho_0^2$. An easy induction on $||\theta(q)||^2$ (together with the previous paragraph and the fact that $C(A)$ is multiplicative) shows the following fact: for any $q \in Q$, there exists $f = f^q \in C(A)$ with $\text{supp}(qf) \subseteq C_{\sqrt{k}}$, i.e. $qf \in RC_{\sqrt{k}}$. But $f \in C(A)$ is saying precisely that $fa = a \quad \forall a \in A$. It follows that $qa = qf^q a \in RC_{\sqrt{k}a}$ for each $q \in Q$ and $a \in A$. In particular (by taking $R$-linear spans), if $A \subseteq A$ is a finite set generating $A$ over $RQ$, then it also generates $A$ over $RC_{\sqrt{k}}$. But $C_{\sqrt{k}}$ is finite, hence $C_{\sqrt{k}}A$ is a finite set generating $A$ as an $R$-module, as required. \hfill \Box 

We will refer to Theorem 4.4 in Chapter 5, where it will translate to a group-theoretic result.

### 4.3 Relation between Bieri–Strebel invariants and tropical varieties

This section concentrates on the main result of the essay, namely, that the set-theoretic complement $\Sigma_A := S(Q) \setminus \Sigma_A$ of a Bieri–Strebel invariant is a union of (projections of) tropical varieties. As earlier, we let $R$ be an integral domain and, as in Chapter 3, we let $K$ be a fixed algebraic closure of the fraction field of $R$. We let $A$ be a left $R[\mathbb{X}^{\pm}]$-module, and $I = \text{Ann}_{RQ}(A)$ its annihilator. We let $\overline{T} = I \otimes_R K$ be the ideal of $K[\mathbb{X}^{\pm}]$ generated by $I$. 

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Note that by definitions we have $\Sigma_A \subseteq S(Q)$ and $\mathcal{T}_v(\overline{I}) \subseteq \mathbb{R}^n$ for a valuation $v$. Thus, to show equivalence, we will need to map $\Sigma_A$ and $\mathcal{T}_v(\overline{I})$ to the same space. For this, $\Sigma_A$ is identified with a subset of $S^{n-1} \subseteq \mathbb{R}^n$ (which we also call $\Sigma_A$ with a slight abuse of notation) via the homeomorphism $S(Q) \cong S^{n-1}$, i.e. we set

$$\Sigma_A = \left\{ \frac{\chi}{\|\chi\|} \mid [\chi] \in \Sigma_A \right\} \subseteq S^{n-1}$$

where $\chi \in \mathbb{R}^n$ and $\chi : Q \to \mathbb{R}$ are related by $\langle \chi, \theta(q) \rangle = \chi(q) \ \forall q \in Q$. We also define a function $\rho$ from the power set of $\mathbb{R}^n$ to the power set of $S^{n-1}$ by “projecting”, i.e. we set

$$\rho(T) := \left\{ \frac{\chi}{\|\chi\|} \mid \chi \in T, \chi \neq 0 \right\} \subseteq S^{n-1} \text{ for } T \subseteq \mathbb{R}^n.$$

This allows us to compare $\Sigma_A$ and $\rho(\mathcal{T}_v(\overline{I}))$ in $S^{n-1}$.

We now assume that $R$ is a Dedekind domain and that $R$ has infinitely many prime ideals. Note that this includes the case $R = \mathbb{Z}$, which will be assumed in Chapter 5, where the results are applied in the context of group theory. This allows us to state our main result:

**Theorem 4.5.** Let $R$ be a Dedekind domain which has infinitely many prime ideals, and let $A$ be a finitely generated $R[\mathbf{X}^\pm]$-module, with annihilator $I = \text{Ann}_{R[\mathbf{X}^\pm]}(A)$. Suppose that $A$ is torsion-free as an $R$-module, i.e. $ra \neq 0$ whenever $0 \neq r \in R$ and $0 \neq a \in A$. Then

$$\Sigma_A^C = \bigcup_{\mathfrak{p} \subseteq R \text{ prime}} \rho \left( \mathcal{T}_v \left( I \otimes_R K \right) \right)$$

where $K$ is a fixed algebraic closure of the fraction field of $R$.

The rest of this section will be used to prove Theorem 4.5. Hence all the valuations in the rest of this section are assumed to satisfy the condition given in Theorem 4.5.

The inclusion ($\supseteq$) in the Theorem is easy: indeed, it follows from

**Proposition 4.6.** Let $R$ be an integral domain and let $A$ be a finitely generated $R[\mathbf{X}^\pm]$-module, with annihilator $I = \text{Ann}_{R[\mathbf{X}^\pm]}(A)$. Let $v$ be a valuation of the algebraic closure $K'$ of $K := \text{Frac}(R)$ with $v(r) \geq 0 \ \forall r \in R$. Then

$$\Sigma_A \cap \rho \left( \mathcal{T}_v \left( I \otimes_R K \right) \right) = \emptyset.$$

**Proof.** We will use Proposition 4.2 for the proof. We let $\overline{I} = I \otimes_R K$.

Note that $\Sigma_A$ is an open set by Proposition 4.2 (as it is a union of open sets), and so $\Sigma_A^C$ is closed. Thus, given that $\rho \left( \mathcal{T}_v(\overline{I}) \right)$ is contained in $\Sigma_A^C$, so is its closure, and this closure must contain $\rho \left( \mathcal{T}_v(\overline{I}) \right) \cap \Sigma_A = \emptyset$.

Thus, let $a \in (K^\times)^n$ be an element with $f(a) = 0$ for any $f \in \overline{I}$. Let $\chi = v(a) \in \mathbb{R}^n$ and suppose w.l.o.g. that $\chi \neq 0$. We must show that $\chi/\|\chi\| \in \Sigma_A^C$.

Suppose for contradiction that $\chi/\|\chi\| \in \Sigma_A$, and let $f \in I$ be an element with

$$\langle \chi, \theta(\text{supp}(f + 1)) \rangle = \text{wt}_\chi(f + 1) > 0,$$
which exists by Proposition 4.2. Then by our assumptions we have \( f_0 = -1 \), and \( f_\alpha = 0 \) whenever \( \langle \chi, \alpha \rangle \leq 0 \) and \( \alpha \neq 0 \).

Consider now the equality \( f(a) = 0 \). We have
\[
\infty = v(0) = v(f(a)) = \min \{ v(f_\alpha a^\alpha) \mid \alpha \in \mathbb{Z}^n \} = \min \{ v(f_\alpha) + \langle v(a), \alpha \rangle \mid \alpha \in \mathbb{Z}^n \}
\]
and so the minimum is attained at least twice on the right-hand side. In particular, the minimum cannot be attained uniquely at \( \alpha = 0 \), hence
\[
v(-1) = v(f_0) \geq v(f_\alpha) + \langle v(a), \alpha \rangle = v(f_\alpha) + \langle \chi, \alpha \rangle
\]
for some \( \alpha \neq 0 \). Thus \( f_\alpha \neq 0 \) for such a value of \( \alpha \) and so \( \langle \chi, \alpha \rangle > 0 \) by the choice of \( f \).

Also, as \( f_\alpha \in R \) we have \( v(f_\alpha) \geq 0 \) by the assumption on \( v \). It follows that \( v(-1) > 0 \).

But we have \( 2v(-1) = v(1) = 2v(1) \) by axiom (ii) of valuations, thus \( v(-1) = v(1) = 0 \).
This gives a contradiction – hence indeed \( \chi/\|\chi\| \in \Sigma_A^C \), as required. \( \Box \)

For proving the inclusion \(( \subseteq \), we will use the assumption that \( R \) is a Dedekind domain with infinitely many primes, as well as the fact that \( A \) is torsion-free over \( R \).

Now let \( \chi \in S^{n-1} \) and suppose that \( \chi \notin \Sigma_A \). For any \( f \in R[\mathbb{X}^\pm] \) such that \( \text{wt}_\chi^0(f) = 0 \), we may define the integral initial form of \( f \) to be
\[
\text{in}_\chi^0(f) := \sum_{\chi(\alpha) = 0} f_\alpha \mathbb{X}^\alpha;
\]
this definition is different from, but resembles the previous definition of initial forms if we take the valuation to be trivial. Then the set
\[
\text{in}_\chi^0(I) := \{ \text{in}_\chi^0(f) \mid f \in I, \text{wt}_\chi^0(f) = 0 \} \cup \{0\}
\]
is clearly an ideal in the group ring \( R(\chi) := R[\mathbb{X}^\alpha : \langle \chi, \alpha \rangle = 0] \). In particular, since \( R \)

is a subring of \( R(\chi) \), the set \( I_R := R \cap \text{in}_\chi^0(I) \) is an ideal in \( R \).

Note that \( \chi \notin \Sigma_A \) means precisely that \( I_R \) does not contain \(-1\), so it is in fact a proper ideal of \( R \). Thus \( I_R \) is contained in some maximal (and therefore prime) ideal \( p \subseteq R \).

Now since \( R \) is a Dedekind domain, we can localise \( R \) at \( p \) to obtain a discrete valuation ring \( R_p \); let \( p \) be its uniformiser, w.l.o.g. \( p \in R \). Thus every non-zero element \( a \) of \( K' := \text{Frac}(R) \) can be uniquely expressed as \( a = p^m r/s \) for some \( m \in \mathbb{Z} \) and \( r, s \in R \setminus p \), and \( K' \) is equipped with \( p \)-adic valuation given by \( v_p(a) = m \).

Now consider \( I' = I \otimes_R K' \subseteq K'[\mathbb{X}^\pm] \): it is easy to see that \( I' \) is precisely the set of elements \( f/r \) where \( f \in I \) and \( r \in R \setminus \{0\} \). We claim that if \( g(X) = 1/\sum_{\alpha \in \mathbb{Z}^n} f_\alpha \mathbb{X}^\alpha \in I' \)

satisfies \( v_p(f_\alpha) \geq 1 \forall \alpha \in \text{supp}(g) \) (where \( f_\alpha \in R \) and \( r \in R \setminus \{0\} \)), then \( up^{-1} g \in I \)

for some \( u \in R \setminus p \). Indeed, \( v_p(f_\alpha) \geq 1 \) means that \( f_\alpha = ps_\alpha/t_\alpha \) for some \( s_\alpha \in R \) and \( t_\alpha \in R \setminus p \), hence we have \( r \prod_{\alpha \in \text{supp}(g)} t_\alpha p^{-1} g \in I \), as claimed.

Note that, since \( \text{in}_\chi^0(I) \) is invariant under multiplication by monomials \( \mathbb{X}^\alpha \) such that \( \langle \chi, \alpha \rangle = 0 \), \( I_R \subseteq p \) means that any monomial contained in \( \text{in}_\chi^0(I) \) must have its coefficient
in \( p \). Let \( f \in I' \). By the previous paragraph (and an easy induction on \( \min \{ \nu_p(f_\alpha) \mid \alpha \in \mathbb{Z}^n \} \)), we obtain an important property of \( \chi \):

If the set \( \{ \langle \chi, \alpha \rangle \mid \alpha \in \text{supp}(f) \} \) attains its minimal value only once (at \( \beta \in \mathbb{Z}^n \), say) then the set \( \{ \nu_p(f_\alpha) \mid \alpha \in \text{supp}(f) \} \) cannot attain its minimal value at the same point \( \beta \).

We will represent a Laurent polynomial \( f \in K'[X^\pm] \) by its \( \chi \)-diagram consisting of points of the form \( (\langle \chi, \alpha \rangle, \nu_p(f_\alpha)) \) (with \( \alpha \in \text{supp}(f) \)) in the plane \( \mathbb{R}^2 \), where the multiplicities of points are counted, i.e. for a point \( (X, Y) \) in the \( \chi \)-diagram of \( f \), the number of \( \alpha \in \text{supp}(f) \) such that \( (X, Y) = (\langle \chi, \alpha \rangle, \nu_p(f_\alpha)) \) is noted. Thus the \( \chi \)-diagram of a polynomial is the projection of the points defining its Newton polytope in \( \mathbb{R}^{n+1} \), obtained by mapping \( (\alpha, \nu_p(f_\alpha)) \in \mathbb{R}^{n+1} \) to \( (\langle \chi, \alpha \rangle, \nu_p(f_\alpha)) \in \mathbb{R}^2 \). As an important particular case, (**) implies that \( I' \) has no elements whose \( \chi \)-diagram is of the following shape.

![Figure 4.1: The “prohibited shape” for a \( \chi \)-graph of a polynomial in \( I' \).](image)

Note that, as long as the elements of \( I' \) are considered, the exact placement of \( X \)- and \( Y \)-axes is inessential. Indeed, horizontal shift of the \( \chi \)-diagram of \( f \) corresponds to multiplying \( f \) by a monomial \( X^\alpha \), and vertical shift corresponds to multiplying \( f \) by a power of \( p \), but \( I' \) is invariant under both these operations.

Having introduced \( \chi \)-diagrams (as well as tropical bases in Chapter 3), we may now prove the following:

**Proposition 4.7.** Let \( R, I, \chi \) and \( p \) be as above. Then there exists a real constant \( c \in [0, \infty) \) such that for all \( f \in I' \), we have \( \text{wt}^\chi_{cvp}(f, \alpha) = \text{wt}^\chi_{cvp}(f) \) for at least two values of \( \alpha \in \text{supp}(f) \).
Proof. The proof is a little bit technical. It involves adding $K'$-multiples of a polynomial $g \in I'$ to a polynomial $f \in I'$ in order to “cancel” points in the $\chi$-graph of $f$ with a “corner” point in the $\chi$-graph of $g$.

We suppose for contradiction that the conclusion is not true. Then for each $c \in [0, \infty)$, we can pick $f_c(X) = \sum_{\alpha \in \mathbb{Z}^n} f_{c, \alpha} X^\alpha \in I'$ and $\alpha_c \in \text{supp}(f_c)$ such that

$$cv_p(f_{c, \alpha_c}) + \langle \chi, \alpha_c \rangle < cv_p(f_{c, \beta}) + \langle \chi, \beta \rangle$$

for each $\beta \in \text{supp}(f_c)$ with $\beta \neq \alpha_c$. Now we set

$$A_c := \{d \in [0, \infty) \mid dv_p(f_{c, \alpha_c}) + \langle \chi, \alpha_c \rangle < dv_p(f_{c, \beta}) + \langle \chi, \beta \rangle \forall \beta \in \text{supp}(f_c) \setminus \{\alpha_c\}\}.$$

Clearly $A_c$ is a open subset of $[0, \infty)$ containing $c$, so it is clear that $\bigcup_{c \in [0, \infty]} A_c = [0, \infty)$. We claim that there exists a finite subset $C \subseteq [0, \infty)$ such that $\bigcup_{c \in C} A_c = [0, \infty)$.

Indeed, for $c \in (0, \infty)$, note that $wt^{X_p}_c = c wt^{X_{\chi}}_{c^{-1}}$ as valuations of $K'[X^\pm]$. Let $G$ be a tropical basis of $I' \otimes_{K'} K$ with respect to the extension of $v_p$ to $K$. Since $G$ is finite and each polynomial in $G$ is a $K$-linear combination of finitely many elements of $I'$, there is a finite subset $G' \subseteq I'$ such that each element of $G$ is a $K$-linear combination of elements of $G'$. Thus, using Corollary 3.6, we can see that the statement “for all $f \in I'$, we have $wt^{X_{\chi}}_{c^{-1}}(f, \alpha) = wt^{X_{\chi}}_{c^{-1}}(f)$ for at least two values of $\alpha \in \text{supp}(f)$” is equivalent to the same statement with $I'$ replaced by $G'$.

Thus we can w.l.o.g. assume that $f_c \in G'$ for each $c \in (0, \infty)$. But then there are only finitely many choices for the pairs $(f_c, \alpha_c)$, and so only finitely many distinct sets $A_c$: $A_1', \ldots, A_N'$, say. These sets form an open cover of $(0, \infty)$, and so together with $A_0' = A_0$, they form an open cover of $[0, \infty)$. This proves the claim.

But also, clearly each $A_i'$ is connected, so it follows that w.l.o.g. we have $A'_0 = [0, d_0')$ and $A'_i = (d_i, d_i')$ for $i = 1, \ldots, N$ (where $d_N' = \infty$), and we have $0 < d_i < d_{i-1}'$ for $i = 1, \ldots, N$.

Now pick $c_i \in A_i'$ for each $i$, and let $(f_{c_i}', \alpha_{c_i}') = (f_{c_i}, \alpha_{c_i})$, with w.l.o.g. $c_0 = 0$. Then the $f_i'$ have $\chi$-graphs of shapes (with inward normals of bounding rays noted)

Now assume that $N$ is minimal with the above construction.
We claim that $N = 0$. This will complete the proof: indeed, if $N = 0$ then the $\chi$-graph of $f'_N = f'_0$ will have the “prohibited shape” as in Figure 4.1, which is not possible. This will contradict the assumption that the conclusion of this Proposition is not true.

Suppose for contradiction that $N \geq 1$. We will show that in fact the pair $(\alpha'_N, f'_N)$ can be chosen in such a way that $A'_N \supseteq (d_{N-1}, \infty)$. This will show that the set $A'_{N-1}$ can be discarded from our open cover of $[0, \infty)$, thus contradicting the minimality of $N$.

Note that we have $d_N < d'_{N-1}$, so $d'_{N-1} \in A'_0$. Thus, by multiplying $f'_N$ by a polynomial in $K'[X^\pm]$, it is possible to “cancel” a point in the $\chi$-graph of $f'_N$ without creating any additional points whose $(d'_{N-1}\chi)$-weight with respect to $v_p$ is not greater than that of the point corresponding to $\alpha'_N$. In particular, for any $\beta \in \text{supp}(f'_N) \setminus \{\alpha'_N\}$, consider the $\chi$-graphs of the polynomials $f'_N(X)$ and $(1 - f_{c_n, \beta}/f_{c_n, \alpha'_N})f'_N(X)$: they have the same points in the region $\{(X, Y) \mid Y + d'_{N-1}X \leq \text{wt}_{d'_{N-1}\chi}(f'_N, \beta)\}$ except for the point $((\chi, \beta), v_p(f_{c_n, \beta})), whose multiplicity in the $\chi$-graphs of $f'_N(X)$ and $(1 - f_{c_n, \beta}/f_{c_n, \alpha'_N})f'_N(X)$ differ by 1, multiplicity of the former one being larger.

Now we wish use this procedure to cancel all the points in the $\chi$-graph of $f_0$ in the strip $C_{D,D'} := \{(X, Y) \mid D \leq (Y + d'_{N-1}X) - \text{wt}_{d'_{N-1}\chi}(f'_N, \alpha'_N) < D'\}$ for arbitrary reals $D' > D > 0$. We may cancel the points in ascending order in their $(d'_{N-1}\chi)$-weights to make sure that no new points are created in the already cancelled region. However, we need to know that this process terminates in finitely many steps. But let

$$\varepsilon := \min\{\text{wt}_{d'_{N-1}\chi}(f'_N, \beta) - \text{wt}_{d'_{N-1}\chi}(f'_N, \alpha'_N) \mid \beta \in \text{supp}(f'_N) \setminus \{\alpha'_N\}\}.$$

Then we have $\varepsilon > 0$, and so for any $D > 0$, no new points are created in the strip $C_{D,D+\varepsilon}$ while cancelling points in this strip. Thus in finitely many steps, we may cancel points in $C_{D,D'}$ by inductively cancelling points in strips $C_{D,D+\varepsilon}, C_{D+\varepsilon,D+2\varepsilon}, \ldots$.

In particular, if we take $D = \varepsilon$, then we may w.l.o.g. assume (by replacing $f'_N$ with its $K'[X^\pm]$-multiple) that the $\chi$-graph of $f'_N$ has the following shape.

Note, in particular, that the $\chi$-graph remains above the lines with normals $(1 : d_N)$ and $(0 : 1)$ in the picture above: indeed, since $v_p(a + b) \geq \min\{v_p(a), v_p(b)\}$ for any $a, b \in K'$, no point in the $\chi$-graph falls below these lines during the procedure of cancellation.
Now we will assume that $N \geq 2$ (the case $N = 1$ is very similar and the same geometric arguments work for it as well). Then we have $d_{N-1} < d'_{N-2}$; let $d'' \in (d_{N-1}, d'_{N-2})$. We may implement the same cancellation procedure on $f_0$ as above, but with $(d''-1)$-weights instead of $(d'_{N-1})$-weights and adding multiples of $f'_{N-1}$ instead of multiples of $f'_N$. We wish to cancel all the points in the green region below, so that all the points in the $\chi$-graph would reside in the blue region.

In the figure, the red dotted outline denotes the shape of the $\chi$-graph of $f'_{N-1}$, and the red circle denotes the point in its $\chi$-graph with minimal $Y$-coordinate. Thus it is possible to add a $K'[X^\pm]$-multiple of $f'_{N-1}$ to $f'_N$ to make the $\chi$-graph reside in the blue region above whenever the point denoted by the red circle has $Y$-coordinate not less than $v_p(f_{cN}, \alpha'_{N})$. But $D' > 0$ was arbitrary, so if we take (see the picture above)

$$D' := D'' = v_p(f_{cN}, \alpha'_{N}) \ominus \min \{ v_p(f_{cN}, \beta) \mid \beta \in \text{supp}(f'_{N-1}) \},$$

then clearly the point denoted by the red circle lies in the blue region.

Since we took $d'' < d'_{N-2}$, this shows that it is possible to choose the pair $(f'_N, \alpha'_N)$ such that $A'_{N} \cup A'_{N-2} \supseteq A'_{N-1} \cup A'_{N-2}$ (and $A'_1 = [0, \infty)$ in the case $N = 1$). This contradicts minimality of $N$, and hence the assumption that $N \geq 1$ is false. Thus $N = 0$, concluding the proof.

Thus if $\chi \notin \Sigma_A$, then the Proposition shows that there exists a prime ideal $p \subseteq R$ and a constant $c \in [0, \infty)$ such that for all $f \in I'$, we have $\text{wt}_{\chi}^v(f, \alpha) = \text{wt}_{\chi}^v(f)$ for at least two values of $\alpha \in \text{supp}(f)$. As every $f \in \mathcal{I} := I' \otimes_{K'} K$ is of the form $f = \sum a_i f_i$ for some $f_i \in I'$ and some $a_i \in K$ which are linearly independent over $K'$, it follows that the statement above also holds for any $f \in \mathcal{I}$.

If $c \neq 0$, then together with Lemma 3.4 and Theorem 3.8 this implies that $c^{-1} \chi \in \mathcal{T}_v(\mathcal{I})$ and so $\chi \in \bigcup_p \rho(\mathcal{T}_v(\mathcal{I}))$, as required.

Hence we may assume that $c = 0$, i.e. for each $f \in \mathcal{I}$, we have

$$\langle \chi, \alpha \rangle = \langle \chi, \text{supp}(f) \rangle \quad \text{for at least 2 distinct } \alpha \in \text{supp}(f).$$

Then Corollary 3.10 implies that $\chi$ is an element of the Bergman fan $\tilde{B}(\mathcal{I})$ of $\mathcal{I}$ with respect to any non-trivial valuation $v : K \to \mathbb{R}_{\infty}$. In particular, $\chi \in \bigcap_{g \in G} S_0(g)$ where
\( G \subseteq I \) is the tropical basis of \( I \) with respect to any non-trivial valuation. It follows that:

For each \( g \in G \), there exist \( r = r_g \geq 2 \) and \( \alpha_{g,1}, \ldots, \alpha_{g,r} \in \text{supp}(g) \) such that \( \text{wt}_\chi(g, \alpha_{g,1}) = \cdots = \text{wt}_\chi(g, \alpha_{g,r}) < \text{wt}_\chi(g, \beta) \) for any \( \beta \in \text{supp}(g) \setminus \{ \alpha_{g,1}, \ldots, \alpha_{g,r} \} \).

But as the Bergman fan is independent of the non-trivial valuation chosen, (†) holds for every non-trivial valuation \( v \) even if we fix the finite subset \( G \subseteq I \).

This is where we use the fact that \( R \) has infinitely many prime ideals. Note that for any \( a \in K' \), we have \( v_p(a) \neq 0 \) for only finitely many primes – precisely those primes that appear in the factorisation of the fractional ideal \( aR \) of \( R \). The way valuations \( v_p \) of \( K' \) are extended to valuations of \( K \) then implies that for any \( a \in K \), we have \( v_p(a) \neq 0 \) for only finitely many primes \( p \subseteq R \). So if we write \( g = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha X^\alpha \in G \), then there are only finitely many primes \( p \subseteq R \) for which the valuation \( v_p \) does not vanish on the finite set

\[ C := \{ g_\alpha | g \in G, \alpha \in \text{supp}(g) \} \]

Thus we may pick a prime \( p \subseteq R \) such that \( v_p(a) = 0 \) for any \( a \in C \). Then (†) implies that \( T_{v_p}(I) \) contains a ray of slope \( \chi \), and furthermore this ray is contained in a linear (as opposed to affine) 1-dimensional subspace of \( \mathbb{R}^n \). That is, \( \{ d'\chi | d' \in [d, \infty) \} \subseteq T_{v_p}(I) \) for some \( d \in \mathbb{R} \). It follows that indeed \( d\chi \in T_{v_p}(I) \) for \( d > 0 \) large enough, hence \( \chi \in \bigcup_p \rho(T_{v_p}(I)) \), as required.

This concludes the proof of Theorem 4.5. \( \square \)

Finally, note that the assumption that \( R \) contains infinitely many primes cannot be dropped. Indeed, let \( D \) be an arbitrary Dedekind domain, let \( p \subseteq D \) be prime, and let \( R = D_p \) be the localisation of \( D \) at \( p \) – clearly, \( R \) is a Dedekind domain with a unique non-zero prime \( pR \). Then every valuation of \( R \) is a non-negative multiple of the \( p \)-adic valuation, so all tropical varieties are non-negative multiples of one fixed tropical variety.

In particular, consider the tropical line in the plane given by the ideal

\[ I = (X + Y + p) \subseteq R[X^{\pm 1}, Y^{\pm 1}] \]

for some \( p \in \mathfrak{B} \). Then sets \( T_{v_p}(I \otimes_R K) \subseteq \mathbb{R}^2 \) (blue) and \( \bigcup_p \rho(T_{v_p}(I \otimes_R K)) \subseteq S^1 \) (red) are visualised below. However, the latter set is not closed in \( S^1 \), so by Proposition 2.4 it cannot be equal to \( \Sigma_{R[X^{\pm 1}, Y^{\pm 1}]/I} \). Thus Theorem 4.5 does not hold in this case.
4.4 Example: Cyclic submodules of \( \mathbb{Z}Q \)-module \( Q \)

This section provides a simple example of a module for which the theory could be applied. Let \( R = \mathbb{Z} \) and observe that \( \mathbb{Z} \) is indeed a Dedekind domain with infinitely many prime ideals. Let \( Q \cong \mathbb{Z}^n \) and consider an action of the ring \( \mathbb{Z}[X^\pm] \cong \mathbb{Z}Q \) on the additive group \((\mathbb{Q},+)\). For \( i = 1, \ldots, n \), let \( m_i = X_i \cdot 1 \) under this action: then we have \( m_i \in \mathbb{Q}^\times \) (as we have \( m_i^{-1} = X_i^{-1} \cdot 1 \in \mathbb{Q} \)). Note that \( \mathbb{Q} \) cannot be finitely generated over \( \mathbb{Z}Q \): if \( A \) is a finitely generated \( \mathbb{Z}Q \)-submodule of \( Q \), then it is easy to see that \( 1/p \in A \) for only finitely many primes \( p \). However, we may take \( A \) to be the cyclic \( \mathbb{Z}[X^\pm] \)-submodule of \( Q \) generated by \( 1 \in \mathbb{Q} \). Then it is clear that

\[
A = \mathbb{Z}[m_1, \ldots, m_n, m_1^{-1}, \ldots, m_n^{-1}] = \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_n}, \frac{1}{p'_1}, \ldots, \frac{1}{p'_{n}}\right]
\]

where we write \( m_i = p_i/p_i' \) for non-zero \( p_i, p_i' \in \mathbb{Z} \) that are coprime.

Now for each \( i \) we have \( (p'_iX_i - p_i1) \cdot 1 = p_i'm_i - p_i = 0 \), and so (by commutativity of \( \mathbb{Z}[X^\pm] \)) \( p'_iX_i - p_i \) annihilates \( A \). Thus \( p'_iX_i - p_i \) is in the ideal \( I := \text{Ann}_{\mathbb{Z}[X^\pm]}(A) \). It follows that

\[
X_i - m_i = \frac{1}{p_i'}(p'_iX_i - p_i) \in \bar{I} := I \otimes_{\mathbb{Z}} K,
\]

where \( K = \overline{\mathbb{Q}} \) is an algebraic closure of \( \mathbb{Q} \). We claim that

\[
\bar{I} = (X_1 - m_1, \ldots, X_n - m_n).
\]

Indeed, we have shown that the ideal in the right hand side of (4.2) is contained in \( \bar{I} \), but it is already a maximal ideal, and \( \bar{I} \subseteq K[X^\pm] \) must be a proper ideal since the action of \( \mathbb{Z}[X^\pm] \) on \( A \) is torsion-free. This proves the claim.

It follows that the variety

\[
V(\bar{I}) = \{\mathbf{m}\} = \{(m_1, \ldots, m_n)\}
\]

contains one point.

Now we write \( v = v_p \) for the \( p \)-adic valuation \( v_p \) of \( K \) for a prime \( p \in \mathbb{Z} \), and write \( \mathcal{T}_p(\bar{I}) \) for \( \mathcal{T}_p(\bar{I}) \). Thus \( \mathcal{T}_p(\bar{I}) \) contains exactly one point for each prime \( p \). Moreover, it is clear that \( \mathcal{T}_p(\bar{I}) = \{0\} \) for any prime \( p \in \mathbb{Z} \) not dividing any of the \( p_i \) or \( p_i' \) — in particular, \( \mathcal{T}_p(\bar{I}) = \{0\} \) for all but finitely many primes \( p \).

The main result of this essay (Theorem 4.5) then implies that

\[
\Sigma^C_A = \left\{ v_p(\mathbf{m}) \left\| v_p(\mathbf{m}) \right\| \mid \right. \left. p \text{ prime, } v_p(m_i) \neq 0 \text{ for some } i \right\}.
\]

In particular, \( \Sigma^C_A \) is a finite set.

We can now verify Theorem 4.4 in this case. Indeed, we have

\[
\Sigma_A = S(Q) \iff v_p(\mathbf{m}) = 0 \text{ for all primes } p \in \mathbb{Z} \iff m_i = \pm 1 \text{ for each } i \iff A = \mathbb{Z}.
\]
Thus if $\Sigma_A = S(Q)$ then certainly $A$ is finitely generated over $\mathbb{Z}$. Conversely, if $A$ is finitely generated over $\mathbb{Z}$ then we have $A = t\mathbb{Z}$ for some $t \in \mathbb{Q}$ (as all finitely generated subgroups of $\mathbb{Q}$ are cyclic). Let $p$ be a prime. Then the set $\{v_p(a) \mid a \in A\}$ is bounded below, but $v_p(X_i^N \cdot 1) = v_p(m_i^N) = Nv_p(m_i)$ for each $i$ and any $N \in \mathbb{Z}$. Taking $N \in \mathbb{Z}$ with either $N$ or $-N$ sufficiently large ensures that $v_p(m_i) = 0$ for each $i$, and so $\Sigma_A = S(Q)$. This verifies Theorem 4.4.
5 Applications to group theory

This chapter gives a short introduction on the historical motivation of Bieri–Strebel invariants. In particular, we consider applying the theory to finitely generated metabelian groups. It turns out that these are precisely extensions of a finitely generated abelian group $Q$ by a finitely generated $\mathbb{Z}Q$-module $A$, and these are exactly our assumptions on $Q$ and $A$ in Chapter 4 in the case $R = \mathbb{Z}$.

5.1 Finitely generated metabelian groups

Let $R = \mathbb{Z}$ and let $G$ be a group given by the short exact sequence of groups

$$\{e\} \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to \{e\} \quad (5.1)$$

with $A$ and $Q$ abelian. Since $\iota$ is an injective homomorphism, we may regard it as an inclusion: in this case $\pi : G \to Q$ is a surjective group homomorphism with kernel $A$. Hence the short exact sequence (5.1) induces a group isomorphism

$$Q \cong G/A \quad (5.2)$$

up to isomorphism in $A$. In other words, $G$ is an extension of $Q$ by $A$. Conversely, it is easy to see that if $G$ is an extension of $Q$ by $A$ then (5.1) holds with $\iota$ (resp. $\pi$) the inclusion (resp. projection) map.

**Definition 5.1.** A group $G$ is called *metabelian* if it satisfies either of the equivalent conditions (5.1) and (5.2) with $A$ and $Q$ abelian.

We will w.l.o.g. assume throughout this chapter that $Q = G/A$. An important fact is the following:

**Lemma 5.2.** $Q$ acts on $A$ on the left via

$$\tau : Q \times A \to A,$$

$$(gA, a) \mapsto gA \cdot a = gag^{-1} \quad (5.3)$$

for any $g \in G$ and $a \in A$. 
Proof. First of all, \( \tau \) is well-defined because (i) \( gag^{-1} \in A \) for any \( g \in G \) since \( A \leq G \), and (ii) if \( gA = hA \) for some \( g, h \in G \) then \( g^{-1}h \in A \) and so \( (g^{-1}h)^{-1}a(g^{-1}h) = a \) since \( A \) is abelian, hence
\[
ha^{-1} = h(g^{-1}h)^{-1}a(g^{-1}h)h^{-1} = gag^{-1}.
\]
\( \tau \) is also a left group action, since
(i) for any \( a \in A \), we have \( A \cdot a = eae^{-1} = a \), and
(ii) for any \( g, h \in G \) and \( a \in A \),
\[
gA \cdot (hA \cdot a) = g(hah^{-1})g^{-1} = (gh)a(gh)^{-1} = ghA \cdot a.
\]
In particular, Lemma 5.2 shows that the map \( \tau \) in (5.3) makes \( A \) into a left \( \mathbb{Z}Q \)-module.

One of the aims of the essay is to study whether a given metabelian group \( G \) is finitely presented. This is a much harder question to ask than whether \( G \) is finitely generated; in fact, we have

**Proposition 5.3.** \( G \) is a finitely generated group \( \iff \) \( Q \) is a finitely generated group and \( A \) is finitely generated as a \( \mathbb{Z}Q \)-module.

Thus when \( G \) is finitely generated, Proposition 5.3 shows that we may use all the previous theory where \( Q \) is a finitely generated abelian group and \( A \) is a finitely generated \( RQ \)-module, by taking \( R = \mathbb{Z} \).

**Proof of Proposition 5.3.** (\( \Rightarrow \)) Let \( G \) be generated by \( g_1, \ldots, g_m \in G \), say. Then clearly \( Q \) is generated by the elements \( g_1A, \ldots, g_mA \), and so \( Q \) is a finitely generated abelian group. By the Structure Theorem for finitely generated abelian groups, we have
\[
Q \cong \mathbb{Z}^n \times (\mathbb{Z}/a_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/a_k\mathbb{Z})
\]
for some \( n, k \in \mathbb{Z}_{\geq 0} \) and \( a_1, \ldots, a_k \in \mathbb{Z}_{\geq 2} \). Hence we may w.l.o.g. assume that \( Q \) is generated as an abelian group by \( g_1A, \ldots, g_nA \) subject to the relations \((g_{n+i}A)^{a_i} = A\) for \( 1 \leq i \leq k \).

This means that \( Q \) is presented as a quotient of a free group on \( \{g_1A, \ldots, g_{n+k}A\} \) by the relations
\[
\mathcal{R} = \{(g_{n+i}A)^{a_i} | 1 \leq i \leq k\} \cup \{[g_iA, g_jA] | 1 \leq i < j \leq n + k\}.
\]
This means that a quotient of \( G \) by a normal subgroup containing
\[
\mathcal{R}' = \{g_{n+i}^{a_i} | 1 \leq i \leq k\} \cup \{[g_i, g_j] | 1 \leq i < j \leq n + k\}
\]
must be a quotient of \( Q \), and conversely all the elements in \( \mathcal{R}' \) are contained in \( A \), hence \( A = \langle \langle \mathcal{R}' \rangle \rangle^G \). In particular, \( A \) is the normal closure of a finite set, \( \mathcal{R}' = \{r_1, \ldots, r_N\} \), say.
Since $A$ is abelian, this shows that each element $a \in A$ is of the form

$$a = \prod_{i=1}^{N} \prod_{j=1}^{k_i} h_{i,j} r_i^{b_{i,j} h_{i,j}^{-1}}$$

(5.4)

for some $k_1, \ldots, k_N \in \mathbb{Z}_{\geq 0}$, and some $h_{i,j} \in G$ and $b_{i,j} \in \mathbb{Z}$. Viewing $A$ as a $\mathbb{Z}Q$-module induced by the action (5.3), (5.4) tells us that

$$a = \prod_{i=1}^{N} \prod_{j=1}^{k_i} (h_{i,j} A \cdot r_i)^{b_{i,j}} = \prod_{i=1}^{N} \left( \sum_{j=1}^{k_i} b_{i,j} h_{i,j} \right) \cdot r_i$$

(5.5)

where $\sum_{j=1}^{k_i} b_{i,j} h_{i,j}$ is considered as an element of $\mathbb{Z}Q$. Hence $A$ is generated by $\mathcal{R}'$ as a $\mathbb{Z}Q$-module.

($\Leftarrow$) Suppose that $Q$ is generated by $g_1 A, \ldots, g_l A$, and that $A$ is generated by $r_1, \ldots, r_N$ as a $\mathbb{Z}Q$-module. Let $g \in G$ be fixed. Since $Q$ is abelian, we have

$$gA = \prod_{i=1}^{l} g_i^{c_i} A$$

for some $c_1, \ldots, c_l \in \mathbb{Z}$, and so we have

$$a := g^{-1} \left( \prod_{i=1}^{l} g_i^{c_i} \right) \in A.$$

By the assumption on $A$, $a$ is of the form (5.5), and hence of the form (5.4); since the action (5.3) of $Q$ on $A$ by conjugation is well-defined, we may w.l.o.g. assume that $h_{i,j} \in \langle g_1, \ldots, g_l \rangle$. Then (5.4) and the fact that $Q$ is finitely generated show that $a$ belongs to the subgroup of $G$ generated by

$$S = \{ r_i \mid 1 \leq i \leq N \} \cup \{ g_j \mid 1 \leq j \leq l \}.$$

Hence $S$ generates $G$.

Thus, if $G$ is a metabelian extension of $Q$ by $A$, then by Proposition 5.3 we can extract information about $G$ by considering $\Sigma_A = \Sigma_A(Q)$.

In particular, $\Sigma_A$ can give information about whether or not the group $G$ is polycyclic. The group $G$ is called polycyclic if there exists a finite chain of subgroups

$$\{ e \} = G_m \trianglelefteq G_{m-1} \trianglelefteq \cdots \trianglelefteq G_0 = G$$

(5.6)

with each quotient $G_i/G_{i+1}$ cyclic. The following corollary to Theorem 4.4 shows how this can be done:

**Corollary 5.4.** Let $G$ be a metabelian extension of $Q$ by $A$. Then

$$G \text{ is polycyclic } \iff \Sigma_A = S(Q).$$
Proof. Applying Theorem 4.4 in the case $R = \mathbb{Z}$ yields that it is enough to show the following: $G$ is polycyclic if and only if $A$ is finitely generated as a $\mathbb{Z}$-module, i.e. as an abelian group. Indeed, suppose $A$ is a finitely generated abelian group. Then by the Structure Theorem for Finitely Generated Abelian Groups, $A$ is polycyclic, and since $Q$ is also finitely generated, $G$ can be obtained from $A$ via finitely many cyclic extensions, hence $G$ is polycyclic as well. Conversely, if $G$ is polycyclic then (5.6) gives a chain of groups

\[ \{e\} = (G_m \cap A) \unlhd (G_{m-1} \cap A) \unlhd \cdots \unlhd (G_0 \cap A) = A \]

with each quotient cyclic, and since $A$ is abelian this implies that $A$ is finitely generated, as required.

\[ \square \]

5.2 Condition on finite presentability

Our last result shows how the Bieri–Strebel invariant of a $\mathbb{Z}Q$-module $A$ can be used to determine whether a metabelian extension $G$ of $Q$ by $A$ can be finitely presented. Namely, we have

**Theorem 5.5.** Let $Q$ be a finitely generated abelian group and let $A$ be a left $\mathbb{Z}Q$-module. Let $G$ be a metabelian extension of $Q$ by $A$. Then

\[ G \text{ is finitely presentable } \iff \Sigma_A \cup (-\Sigma_A) = S(Q). \]

The full proof of Theorem 5.5 can be found in [3] and is not given here. We only give a sketch proof of the implication $(\Leftarrow)$.

Recall that $Q \cong \tilde{Q} \times \text{Tor}(Q)$ for a torsion-free finitely generated abelian group $\tilde{Q} \cong \mathbb{Z}^n$. We claim that it is enough to prove the Theorem with $Q$ replaced by $\tilde{Q}$. Indeed, it was shown earlier that $\Sigma_A(Q) = \Sigma_A(\tilde{Q})$. Thus it is enough to show that a metabelian extension $G$ of $Q$ by $A$ is finitely presentable if and only if the extension $\tilde{G}$ of $\tilde{Q}$ by $A$ is.

For this, note that $\tilde{G}$ is a normal subgroup of finite index in $G$. Thus we may take the presentation of $\tilde{G}$ and append a (finite) transversal of $\tilde{G}$ in $G$ to the generators and the (finite) multiplication table for this transversal to the relators to obtain a presentation of $G$, which is finite if the presentation of $\tilde{G}$ we started with is finite.

Conversely, given a finite presentation $\langle g_1, \ldots, g_m \mid r_1, \ldots, r_M \rangle$ for $G$, we may use the Reidemeister–Schreier method [1, Section III.6] to obtain a finite presentation of $\tilde{G}$. This consists of taking a transversal $T = \{t_1, \ldots, t_N\}$ of $\tilde{G}$ in $G$ (called a Schreier transversal) with the property that if we write the $t_i$ in terms of the $g_j$, then any initial segment of each $t_i$ is also in $T$. The existence of this transversal is easy to show, e.g. by induction on $N$. As generators of $\tilde{G}$ we then take the elements $t_i g_j t_i^{-1}$ for $1 \leq i \leq N$, $1 \leq j \leq m$ and such $l = l(i, j) \in \{1, \ldots, N\}$ that $t_i g_j \in t_l \tilde{G}$. Finally, it is possible to rewrite the relators $t_i r_j t_i^{-1}$ for $1 \leq i \leq N$ and $1 \leq j \leq M$ in terms of the new generators, and we let these be the relators in the presentation of $\tilde{G}$. Then the group given by the finite presentation obtained is indeed isomorphic to $\tilde{G}$. Thus we lose no generality by assuming that $Q$ is torsion-free, as claimed.
Hence we assume that $Q$ is generated by elements $Q = \{q_1, \ldots, q_n\}$ as a free abelian group, and that the $\mathbb{Z}Q$-module $A$ is generated by a finite set $\mathcal{A}$. W.l.o.g., we may assume that $\mathcal{A}$ contains all the commutators $a_{ij} := [q_i^{-1}, q_j]$ and $b_{ij} := [q_i, q_j]$.

Note that given the left action of $\mathbb{Z}Q$ on $A$, we can equip $A$ with a right $\mathbb{Z}Q$-action by setting $q \cdot a = a \cdot q^{-1}$ for $q \in Q$ and $a \in A$. We denote the resulting right $\mathbb{Z}Q$-module with underlying set $A$ by $A^\ast$. We call $A$ a tame module if for any character $\chi$ of $Q$, there is an element $f \in C(A) \cup C(A^\ast)$ such that $\chi(f) > 0$. Thus by Proposition 4.2, Theorem 5.5 is saying precisely that an extension of $Q$ by $A$ is finitely presentable if and only if the underlying $\mathbb{Z}Q$-module $A$ is tame.

We now sketch the proof of $(\Leftarrow)$ direction of Theorem 5.5. Thus we assume that $A$ is tame, i.e. $\Sigma_A \cup (-\Sigma_A) = S(Q)$. As $S(Q)$ is compact, the open cover of $\Sigma_A \cup (-\Sigma_A)$ given by Proposition 4.2 has a finite subcover, indexed by a finite subset $\mathcal{F} \subseteq C(A) \cup C(A^\ast)$; that is, there is a finite subset $\mathcal{F} \subseteq C(A) \cup C(A^\ast)$ such that

$$S(Q) = \bigcup_{f \in \mathcal{F}} \{ [\chi] \mid \chi(f) > 0 \}.$$

We will need consider elements of $Q$ as given by words in the elements $q_i \in Q$. For this, given an element $q = q_1^{\alpha_1} \cdots q_n^{\alpha_n} \in Q$ (where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$), we denote the element $q_1^{\alpha_1} \cdots q_n^{\alpha_n} \in F(Q)$ by $q$, where $F(Q)$ is the free group on the set $Q$. Furthermore, if $q \in F(Q \cup \mathcal{A})$ then we denote by $\overline{q}$ the element of $F(Q)$ given by mapping all elements of $A$ to $\{e\}$, taking the corresponding element of $Q$, and taking its image under the map $q' \mapsto \overline{q'}$ (as defined in the previous sentence).

We now consider the group $G_\infty$ with presentation given by

- generators $\mathcal{A} \cup Q$;
- relations of five kinds:

\[
\begin{align*}
[q_i, q_j] &= b_{ij} & \text{for all } 1 \leq i < j \leq n, & (R1) \\
[q_i^{-1}, q_j] &= a_{ij} & \text{for all } 1 \leq i, j \leq n, & (R2) \\
[a, \overline{q}^{-1}b\overline{q}] &= e & \text{for all } a \in \mathcal{A}, q \in Q, & (R3) \\
\prod_{q \in \supp(f)} \overline{q}a^f_{q}\overline{q}^{-1} &= a & \text{for all } a \in \mathcal{A}, f = \sum_{q \in Q} f_q q \in \mathcal{F} \cap C(A), & (R4) \\
\prod_{q \in \supp(f)} \overline{q}^{-1}a^f_{q}\overline{q} &= a & \text{for all } a \in \mathcal{A}, f = \sum_{q \in Q} f_q q \in \mathcal{F} \cap C(A^\ast). & (R5)
\end{align*}
\]

Note that there are finitely many relations of the forms (R1), (R2), (R4) and (R5), and (provided $Q$ is infinite) infinitely many relations of the form (R3). (Relations (R2) are only included here to make the proof of the first part of the following lemma easier.)

We also let $A_\infty := \langle \mathcal{A} \rangle^{G_\infty}$ be the normal subgroup generated by $\mathcal{A}$. Then it is easy to see from the given presentation that $G_\infty/A_\infty \cong \mathbb{Z}^n \cong Q$. We aim to show the following:

**Lemma 5.6.** (i) $A_\infty$ is abelian.
(ii) All but finitely many relations of the form \((R3)\) in the presentation of \(G_\infty\) are redundant. In particular, \(G_\infty\) is finitely presentable.

Given Lemma 5.6, the proof that \(G\) is finitely presentable is easily completed. Indeed, note that all the relations \((R2)-(R5)\) are satisfied in \(G\), hence \(G \cong G_\infty/K\) for some normal subgroup \(K \leq G_\infty\), where the isomorphism is given by identity on the generators \(A \cup Q\). Now since the map which fixes \(Q\) and sends \(A\) to \(\{e\}\) induces isomorphisms \(G_\infty/A_\infty \cong Q \cong G/A\), it follows that \(G_\infty/A_\infty\) is a quotient of \(G_\infty/K\), so in particular \(K \leq A_\infty\).

Now part (i) of the Lemma says that \(A_\infty\) is abelian, and so \(G_\infty\) is a metabelian extension of \(Q\) by \(A_\infty\). Hence \(A_\infty\) is a left \(\mathbb{Z}Q\)-module, which is finitely generated since \(A\) is finite. But we have ring isomorphisms

\[
\mathbb{Z}Q \cong \mathbb{Z}[X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}] \cong \mathbb{Z}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_1Y_1-1, \ldots, X_nY_n-1),
\]

so by Hilbert's Basis Theorem and the fact that \(\mathbb{Z}\) is Noetherian it follows that \(\mathbb{Z}Q\) is Noetherian as well. In particular, \(A_\infty\) is a finitely generated module over a Noetherian ring, therefore Noetherian. It follows that the submodule \(K \leq A_\infty\) is finitely generated. (Note, \(K\) is indeed a submodule as it is abelian and normal in \(G_\infty\).)

This is saying precisely that \(K \leq G_\infty\) is the normal closure of a finite set. Thus given a presentation of \(G_\infty\), we may append finitely many relations to it to obtain a presentation of \(G\). But by part (ii) of the Lemma, \(G_\infty\) is finitely presented, hence so is \(G\). This will conclude the proof of the \((\Leftarrow)\) direction of Theorem 5.5.

We now sketch the proof of Lemma 5.6.

For part (i), it is enough to show that \(A_\infty\) is a quotient of an abelian group.

In particular, let \(G'_\infty\) be the group given by the presentation with generators \(A \cup Q\), and relations \((R2)\) and \((R3)\). Let \(A'_\infty := \langle \langle A \rangle \rangle_{G'_\infty}\). Then \(G_\infty\) is the quotient of \(G'_\infty\) by the normal closure of relators given by \((R4)\) and \((R5)\), so \(A_\infty\) is a quotient of \(A'_\infty\). Hence it is enough to show that \(A'_\infty\) is abelian.

For this, it is enough to show that if \(w \in F(A \cup Q)\) is an arbitrary word (and an element of \(G'_\infty\) it represents is also denoted by \(w\) by a slight abuse of notation), and \(a, b \in A\) are arbitrary elements, then \([a, w^{-1}bw] = e\) in \(G'_\infty\). And for this, it is enough to show that \(w^{-1}bw = \overline{w}^{-1}b\overline{w}\) for any \(b \in A\) and \(w \in F(Q \cup A)\), as then the relations \((R3)\) imply that \(A'_\infty\) is abelian.

We call a word \(w \in F(A \cup Q)\) bad if it starts and ends with elements \(q_i^{+1}\) and \(q_j^{+1}\) (respectively) of \(Q \cup Q^{-1}\), and we have \(i > j\). We define the badness of \(w\) to be the pair \((\alpha, \beta)\), where \(\alpha\) is the number of bad subwords of \(w\), and \(\beta\) is the number of letters in \(w\) which are elements of \(A \cup A^{-1}\). We clearly have \(w = \overline{w}\) in \(F(A \cup Q)\) if and only if \(w\) has badness \((0, 0)\).

We will prove that \(w^{-1}bw = \overline{w}^{-1}b\overline{w}\) in \(G'_\infty\) by induction on the badness of \(w\), where badness is equipped with the lexicographic order. Indeed, if the badness of \(w\) is \((0, 0)\) then we are done. Otherwise, we may put \(w\) in the form

\[
w = w'xw''
\]

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where $w', w'' \in F(A \cup Q)$ are words, $w'$ has badness $(0, 0)$ and $x \in A \cup A^{-1} \cup Q \cup Q^{-1}$ is a single letter. We may also require that $w'x$ has badness larger than $(0, 0)$. We have two cases:

- If $x \in A \cup A^{-1}$, then relations (R3) imply that $[x, w'^{-1}bw'] = e$ in $G'_\infty$ and so
  \[ w^{-1}bw = (w'xw'')^{-1}b(w'xw'') = (w'w'')^{-1}b(w'w''). \]

- If $x \in Q \cup Q^{-1}$, then note that the one-letter word $x$ has badness $(0, 0)$, hence the word $w'$ must have at least one letter. The relations (R2) imply
  \[ q_jq_i = q_iq_jq_i, \quad q_jq_i^{-1} = q_i^{-1}q_ja_{ji}, \quad q_j^{-1}q_i = a_{ji}q_iq_j^{-1}, \quad q_j^{-1}q_i^{-1} = q_i^{-1}a_{ji}^{-1}q_j^{-1}. \]

Applying this to $x$ and the last letter of $w'$ implies that $w = w_0$ in $G'_\infty$ for some word $w_0$ which has fewer bad subwords than $w$.

Thus in either case, we have $w^{-1}bw = w_0^{-1}bw_0$ for some word $w_0 \in F(Q \cup A)$ which has smaller badness than $w$. By induction hypothesis, we are done. Thus indeed $A_\infty$ is abelian, proving Lemma 5.6 (i).

For part (ii) of Lemma 5.6, for any real $\rho > 0$ we define the group $G_\rho$, which is given by the same presentation as $G_\infty$ except that we replace the infinite set of relations (R3) by the finite set of relations
\begin{align*}
[a, \bar{q}^{-1}b\bar{q}] = e \quad \text{for all } a, b \in A \text{ and such } q \in Q \text{ that } \|\theta(q)\| < \rho, \quad \text{(R3')} \end{align*}
where as previously, $\theta : Q \to \mathbb{R}^n$ is the map sending the $q_i$ to the standard basis. This justifies the notation $G_\infty$ as the group $G_\rho$ by taking $\rho = \infty$ in (R3').

Note that, as discussed previously, $\|\theta(q)\|^2 \in \mathbb{Z}$ for all $q \in Q$, and so $G_\rho$ is constant for $\rho \in (\sqrt{k}, \sqrt{k} + 1]$ for any $k \in \mathbb{Z}_{\geq 0}$.

Clearly $G_\rho$ is finitely presented for each real $\rho > 0$. So the proof of Lemma 5.6 follows immediately if we can show that $G_\infty = G_\rho$ for a sufficiently large $\rho > 0$. We will sketch a proof of this.

We aim to show that given any $\rho > 2n\rho_0$, we have $G_{\rho + \epsilon} = G_\rho$, i.e. that the relations of the form (R3) with $\rho \leq \|\theta(q)\| < \rho + \epsilon$ follow from the relations (R3'). It then follows by induction that $G_\rho = G_\infty$ for any $\rho > 2n\rho_0$, which will conclude the proof.

Recall that we have a finite subset $F \subseteq C(A) \cup C(A^*)$ with the property that the union
\[ \bigcup_{f \in F} \{[\chi] \in S(Q) \mid (\chi, \text{supp}(f)) > 0 \} \]
is an open cover of $S(Q)$. It follows that (as in the proof of Theorem 4.4) we can use Lemma 4.3 for $F$, again realised as a collection of subsets of $\mathbb{Z}^n$ by identifying elements with their supports. It follows there exist constants $\epsilon, \rho_0 > 0$ that given any $\rho > \rho_0$ and any $y \in B_{\rho + \epsilon}$, there exists $f \in F$ with $y + \text{supp}(f) \subseteq B_\rho$.

Now let $\rho = \sqrt{k} > 2n\rho_0$ for some $k \in \mathbb{Z}$, let $a, b \in A$ and let $q \in Q$ be an element with $\|\theta(q)\| = \rho$. Then it is enough to show that $[a, \bar{q}^{-1}b\bar{q}] = e$ in $G_\rho$. By the previous
paragraph, we have $\theta(q) + \text{supp}(f) = \text{supp}(fq) \subseteq B_\rho$ for some $f \in \mathcal{F}$, and we may assume that $f \in C(A^*)$ (the case $f \in C(A)$ is similar). Using the commutation relation

$$[x, yz] = xyzx^{-1}z^{-1}y^{-1} = xyy^{-1}[x, z]y^{-1} = (xyx^{-1})[x, z](xyx^{-1})^{-1}[x, y]$$

and the relations (R5) we obtain

$$[a, \bar{q}^{-1}b\bar{q}] = \left[ a, \prod_{r \in \text{supp}(f)} (r \cdot \bar{q})^{-1}b^{f_r} (r \cdot \bar{q}) \right] = \prod_{r \in \text{supp}(f)} s_r \left[ a, (r \cdot \bar{q})^{-1}b^{f_r} (r \cdot \bar{q}) \right] s_r^{-1} \quad (5.7)$$

in $G_\rho$ for some suitable elements $s_r \in G_\rho$. But it turns out that, in the given circumstances, all the commutators in the right hand side of (5.7) are conjugate in $G_\rho$ to the same terms with “$\bar{r} \cdot \bar{q}$” replaced by “$(rq)$” – the proof of this fact in [3] is technical and is not given here. Then relations (R3′) show that all these commutators are in fact equal to the identity, hence $[a, \bar{q}^{-1}b\bar{q}] = e$ in $G_\rho$, as required.

This shows Lemma 5.6 (ii) and hence the $(\Leftarrow)$ direction of Theorem 5.5.

5.3 Example: Cyclic submodules of $\mathbb{Z}Q$-module $\mathbb{Q}$ (continued)

Recall the example in the Section 4.4: we take the cyclic $\mathbb{Z}[X^\pm]$-module

$$A = \mathbb{Z} \left[ p_1^{-1}, \ldots, p_n^{-1}, p_1^{-1}, \ldots, p_n^{-1} \right]$$

with the action given by $X_i \cdot 1 = p_i/p'_i$, where for each $i$ the non-zero integers $p_i$ and $p'_i$ are coprime. We let $m_i = p_i/p'_i$ and let $\mathbf{m} = (m_1, \ldots, m_n) \in (\mathbb{Q}^\times)^n$. We showed in Section 4.4 that

$$\Sigma_C^A = \left\{ \frac{v_p(\mathbf{m})}{\|v_p(\mathbf{m})\|} \mid p \text{ prime, } v_p(m_i) \neq 0 \text{ for some } i \right\}.$$

Now suppose in addition that the elements $m_i = p_i/p'_i$ are linearly independent over $\mathbb{Z}$ as elements of the multiplicative group $\mathbb{Q}^\times$, i.e. that there is no non-zero $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ such that $\mathbf{m}^\mathbf{a} = m_1^{a_1} \cdots m_n^{a_n} = 1$. Define the group $\tilde{G}$ by

$$\tilde{G} = \left\{ \begin{pmatrix} 1 & a \\ 0 & \mathbf{m}^\mathbf{a} \end{pmatrix} \mid a \in A, \mathbf{a} \in \mathbb{Z}^n \right\} \leq GL_2(\mathbb{Q});$$

it is easy to see that $\tilde{G}$ is indeed closed under multiplication and so is a group. We also define

$$\tilde{A} = \left\{ \tilde{a} := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in A \right\}$$

and for $i = 1, \ldots, n$, we define

$$\tilde{X}_i = \begin{pmatrix} 1 & 0 \\ 0 & m_i^{-1} \end{pmatrix}.$$
It is clear that $\tilde{A} \leq \tilde{G}$ is a normal subgroup: indeed, $\tilde{A}$ is preserved by conjugation by any upper triangular matrix with coefficients in $A$. Also, it is easy to see that $\tilde{A}$ is abelian and that $G/\tilde{A} \cong \mathbb{Z}^n$: the latter follows from the fact that the $m_i$ are linearly independent in $\mathbb{Q}^\times$.

Finally, the action of the $\tilde{X}_i$ on $\tilde{A}$ given by $\tilde{X}_i \cdot \tilde{a} = \tilde{X}_i a \tilde{X}_i^{-1}$ is the same as the action giving $A$ the structure of a $\mathbb{Z}Q$-module, and so $\tilde{G}$ is a metabelian extension of $A$ by $Q \cong \mathbb{Z}^n$, as required.

Now consider the special case when $m_i \in \mathbb{Z}$ for each $i$. Then for any prime $p \in \mathbb{Z}$, the variety $\mathcal{T}_p(\mathcal{I}) = \{ v_p(m) \}$ is a point in the box $[0, \infty)^n$, and so all points in $\Sigma_A^C \subseteq S^{n-1}$ have non-negative coordinates. It follows that there are no points in $\Sigma_A^C \cap (-\Sigma_A^C)$, as each such point would have all coordinates equal to 0. Hence $\Sigma_A \cup (-\Sigma_A) = S^{n-1}$, i.e. $A$ is tame. Thus Theorem 5.5 implies that the metabelian group $G$ is finitely presented.

We verify this fact by giving an explicit presentation. Let $G$ be a group given by the presentation

$$
G = \langle c, b_1, \ldots, b_n \mid b_i^{-1}c b_i = c^{m_i} \text{ for } 1 \leq i \leq n, \ [b_i, b_j] = e \text{ for } 1 \leq i < j \leq n \rangle. \tag{5.8}
$$

We claim that $G \cong \tilde{G}$. Indeed, by setting

$$
\tilde{c} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \tilde{G}, \quad \tilde{b}_i := \begin{pmatrix} 1 & 0 \\ 0 & m_i \end{pmatrix} \in \tilde{G},
$$

we can see that all relations in (5.8) are satisfied under the map $c \mapsto \tilde{c}, b_i \mapsto \tilde{b}_i$, so this map induces a group homomorphism $\varphi : G \to \tilde{G}$. It is easy to see that $\tilde{c}$ and the $\tilde{b}_i$ generate $\tilde{G}$, hence $\varphi$ is surjective, i.e. $\tilde{G}$ is a quotient of $G$.

Now observe that any element $g \in G$ can be written as

$$
g = b^\alpha c^\beta b^{-\alpha'}
$$

for some $\alpha, \alpha' \in \mathbb{Z}^n$ with $\alpha_i, \alpha'_i \geq 0 \ \forall i$ and some $\beta \in \mathbb{Z}$, where we write $b^\gamma$ for the element $b_1^{\gamma_1} \cdots b_n^{\gamma_n} \in G$: indeed, relations in (5.8) imply that

$$
c^{1} b_i = b_i c^{m_i}, \quad b_i^{-1} c^1 = c^{m_i} b_i^{-1}, \quad b_i^1 \text{ and } b_j^1 \text{ commute},
$$

which can be used to “transport” all positive powers of the $b_i$ to the front and all negative powers to the back of the word representing $g \in G$. Thus we have

$$
\varphi(g) = \varphi(b^\alpha) \varphi(c^\beta) \varphi(b^{-\alpha'}) = \begin{pmatrix} 1 & 0 \\ 0 & m^{\alpha} \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & m^{-\alpha'} \end{pmatrix} = \begin{pmatrix} 1 & m^{-\alpha'} \beta \\ 0 & m^{\alpha-\alpha'} \end{pmatrix}
$$

and so $\varphi(g)$ is the identity if and only if $\beta = 0$ and $\alpha = \alpha'$, if and only if $g = e$ in $G$. Thus $\varphi$ is injective and so $G \cong \tilde{G}$, as claimed.
References


